RECURRENCE VERSUS TRANSIENCE FOR WEIGHT-DEPENDENT RANDOM CONNECTION MODELS

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Abstract. We investigate a large class of random graphs on the points of a Poisson process in d-dimensional space, which combine scale-free degree distributions and long-range effects. Every Poisson point carries an independent random weight and given weight and position of the points we form an edge between two points independently with a probability depending on the two weights and the distance of the points. In dimensions $d \in \{1, 2\}$ we completely characterise recurrence vs transience of random walks on the infinite cluster. In $d \geq 3$ we prove transience in all cases except for a regime where we conjecture that scale-free and long-range effects play no role. Our results are particularly interesting for the special case of the age-dependent random connection model recently introduced in [8].

1. Introduction and statement of results

In this paper we investigate the classical problem of transience versus recurrence of random walks on the infinite cluster of a general class of geometric random graphs in d-dimensional Euclidean space, which we denote as the *weight-dependent random connection model.* This class contains classical models like the Boolean and random connection models as well as models that have long edges and scale-free degree distributions. The focus in this paper is on those instances of graph models where the long-range or scale-free nature of graphs leads to new or even surprising results.

The vertex set of the weight-dependent random connection model is a Poisson process of unit intensity on $\mathbb{R}^d \times [0, 1]$. We think of a Poisson point $\mathbf{x} = (x, s)$ as a *vertex* at *position* x with *weight* s^{-1}. Two vertices $\mathbf{x} = (x, s)$ and $\mathbf{y} = (y, t)$ are connected by an edge with probability $\varphi(\mathbf{x}, \mathbf{y})$ for a connectivity function

$$\varphi\colon (\mathbb{R}^d \times [0, 1]) \times (\mathbb{R}^d \times [0, 1]) \to [0, 1], \tag{1}$$

satisfying $\varphi(\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{y}, \mathbf{x})$. Connections between different (unordered) pairs of vertices occur independently. We assume throughout that φ has the form

$$\varphi(\mathbf{x}, \mathbf{y}) = \varphi\big((x, s), (y, t)\big) = \rho\big(h(s, t, |x - y|)\big) \tag{2}$$

for a non-increasing, integrable *profile function* $\rho\colon \mathbb{R}_+ \to [0, 1]$ and a suitable *kernel function* $h\colon [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$. We assume further that h is non-increasing in the first two arguments, and non-decreasing in the third argument. Hence vertices whose positions are far apart are less likely to be connected while vertices with large weight are likely to have many connections. To standardise the notation (without losing generality) we assume that

$$\int_{\mathbb{R}^d} \rho(|x|)\, dx = 1. \tag{3}$$

With this convention it is easy to see that the degree distribution of a vertex depends only on the kernel function h, but ρ can have a massive influence on the likelihood of long edges.

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We next give concrete examples for the kernel function h, and demonstrate that our setup yields a number of well-known models in continuum percolation theory. We define the functions in terms of parameters $\gamma \in (0,1)$ and $\beta \in (0,\infty)$. The parameter γ determines the strength of the influence of the vertex weight on the connection probabilities, large γ correspond to strong favouring of vertices with large weight. In particular, except for the first example, the models are scale-free with power-law exponent

$$\tau = 1 + \tfrac{1}{\gamma}.$$

The edge density can be controlled by the parameter β, increasing β increases the expected number of edges connected to a vertex at the origin. Note also that varying β can also be interpreted as rescaling distances and therefore it is equivalent to varying the (spatial) density of the underlying Poisson process.

- We define the *plain kernel* as
$$h^{\text{plain}}(s,t,v) = \frac{1}{\beta} v^d. \tag{4}$$
In this case we have no dependence on the weights. If $\rho(r) = 1_{[0,a]}$ for $a = \sqrt[d]{d/\omega_d}$ and ω_d is the area of the unit sphere in $\mathbb{R}^d$, this gives the *Gilbert disc model* with radius $\sqrt[d]{\beta a}$. Functions ρ of more general form lead to the (ordinary) *random connection model*, including in particular a continuum version of *long-range percolation* when ρ has polynomial decay at infinity.
- We define the *sum kernel* as
$$h^{\text{sum}}(s,t,v) = \frac{1}{\beta} \left(s^{-\gamma} + t^{-\gamma}\right)^{-1} v^d. \tag{5}$$
Interpreting $(\beta a s^{-\gamma})^{1/d}, (\beta a t^{-\gamma})^{1/d}$ as random radii and letting $\rho(r) = 1_{[0,a]}$ we get the *Boolean model* in which two vertices are connected by an edge if the associated balls intersect. We get a further variant of the model with the *min-kernel* defined as
$$h^{\min}(s,t,v) = \frac{1}{\beta}(s \wedge t)^{\gamma} v^d.$$
Because $h^{\text{sum}} \le h^{\min} \le 2h^{\text{sum}}$ the two kernels show qualitatively the same behaviour.
- For the *max-kernel* defined as
$$h^{\max}(s,t,v) = \frac{1}{\beta}(s \vee t)^{1+\gamma} v^d,$$
we may choose any $\gamma > 0$. This is a continuum version and generalization of the ultra-small scale-free geometric networks of Yukich [27], which is also parametrized to have power-law exponent $\tau = 1 + \frac{1}{\gamma}$.
- A particularly interesting case is the *product kernel*
$$h^{\text{prod}}(s,t,v) = \frac{1}{\beta} s^{\gamma} t^{\gamma} v^d, \tag{6}$$
which leads to a continuum version of the *scale-free percolation* model of Deijfen et al. [4, 11], see also [5]. This model combines features of scale-free random graphs and polynomial-decay long-range percolation models (for suitable choice of ρ).
- Our final example for h is the *preferential attachment kernel*
$$h^{\text{pa}}(s,t,v) = \frac{1}{\beta}(s \vee t)^{1-\gamma}(s \wedge t)^{\gamma} v^d, \tag{7}$$
which gives rise to the *age-dependent random connection model* introduced by Gracar et al. [8] as an approximation to the local weak limit of the spatial preferential attachment model in Jacob and Mörters [14]. In this model, s and t actually play the role of the birth times of vertices in the underlying dynamic network, we therefore refer to vertices with small s, equivalently with large weight, as old vertices. This model also combines

scale-free degree distributions with power-law exponent $\tau = 1 + \frac{1}{\gamma}$ and long edges in a natural way, but has a fundamentally different graph topology, as we will see.

The weight-dependent random connection model with its different kernels has been studied in the literature under various names, we summarize some of them in Table 1.

Table 1. Terminology of the models in the literature.

Vertices	Profile	Kernel	Name and reference
Poisson	indicator	plain	Random geometric graph, Gilbert disc model [24]
Poisson	general	plain	Random connection model [18]
			Soft random geometric graph [23]
lattice	polynomial	plain	Long-range percolation [1]
Poisson	indicator	sum	Boolean model [9, 19]
lattice	indicator	max	Ultra-small scale-free geometric networks [27]
Poisson	indicator	min	Scale-free Gilbert graph [12]
lattice	polynomial	prod	Scale-free percolation [4, 11]
Poisson	polynomial	prod	Inhomogeneous long-range percolation [5]
			Continuum scale-free percolation [6]
Poisson	general	prod	Geometric inhomogeneous random graphs [2]
Poisson	general	pa	Age-dependent random connection model [8]

We now focus on a profile function with polynomial decay

$$\lim_{v\to\infty} \rho(v)\, v^{\delta} = 1 \quad \text{for a parameter } \delta > 1, \tag{8}$$

and fix one of the kernel functions described above. We keep γ, δ fixed and study the resulting graph $\mathcal{G}^\beta$ as a function of β. Note that our assumptions $\delta > 1$ and $\gamma < 1$ guarantee that $\mathcal{G}^\beta$ is locally finite for all values of β, cf. [8, p.8]. We informally define β_c as the infimum over all values of β such that $\mathcal{G}^\beta$ contains an infinite subgraph (henceforth the infinite *cluster*); for a rigorous definition we refer to Section 2. If $d \geq 2$, we always have $\beta_c < \infty$, cf. [11]. General arguments in [7] yield that there is at most one infinite subgraph of $\mathcal{G}^\beta$, and hence there is a unique infinite subgraph whenever $\beta > \beta_c$. We study the properties of this infinite cluster.

Two cases correspond to different network topologies. If $\gamma > \frac{1}{2}$ for the product kernel, or any $\gamma > 0$ for the max kernel, or $\gamma > \frac{\delta}{\delta+1}$ for the preferential attachment, sum, or min kernel, we have $\beta_c = 0$, i.e. there exists an infinite cluster irrespective of the edge density, see [11, 27, 15]. We say that this is the *robust case*. Otherwise, if $\gamma < \frac{1}{2}$ for product or $\gamma < \frac{\delta}{\delta+1}$ for the preferential attachment, sum or min kernel, we say we are in the *non-robust case*.

Our main interest is whether the infinite cluster is recurrent (i.e., whether simple random walk in the cluster returns to the starting point almost surely), or transient (i.e., simple random walk on the cluster has positive probability of never returning to the starting point). Our results are summarized in the following theorem.

Theorem 1.1 (Recurrence vs. transience of the weight-dependent random connection model)**.** *Consider the weight-dependent random connection model with profile function* (8).

(a) For preferential attachment kernel, sum kernel, or min kernel and $\beta > \beta_c$, the infinite component is
- transient *if either* $1 < \delta < 2$ *or* $\gamma > \delta/(\delta + 1)$*;*
- recurrent *if* $d \in \{1, 2\}$*,* $\delta > 2$ *and if* $\gamma < \delta/(\delta + 1)$*.*

(b) For the product kernel and $\beta > \beta_c$, the infinite component is
- transient *if either* $1 < \delta < 2$ *or* $\gamma > 1/2$*;*
- recurrent *if* $d \in \{1, 2\}$*,* $\delta > 2$ *and if* $\gamma < 1/2$*.*

(c) *For the max kernel and $\beta > \beta_c$, the infinite component is* transient.

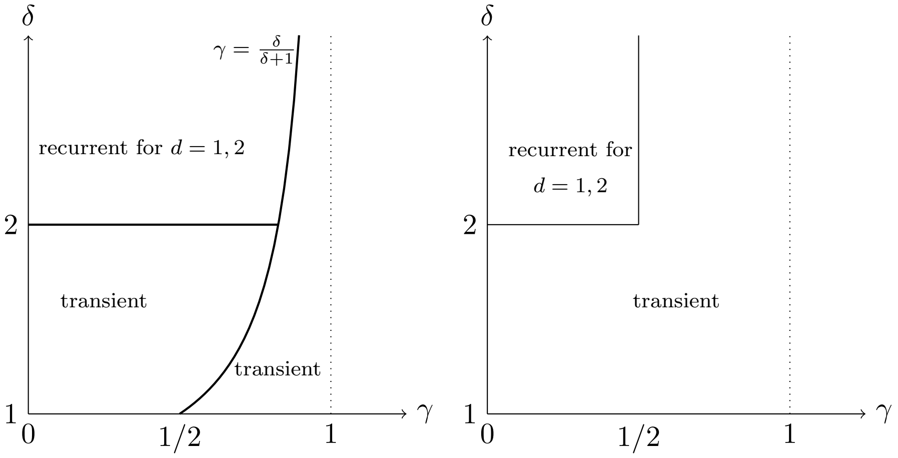

FIGURE 1. The different phases in Theorem 1.1: Left: preferential attachment, sum and min kernels. Right: product kernel.

Remarks:

- Loosely speaking, for the models in (a) and (b) the walk can travel to infinity using long edges if there are enough of them, i.e. if $\delta < 2$. For the models in (a) the walk can also use that vertices of ever increasing weight can be reached using a pool of intermediate vertices, which is big enough if $\delta < \frac{\gamma}{1-\gamma}$. For the model in (b) with $\gamma > \frac{1}{2}$ and the model in (c) with $0 < \gamma < 1$ the walk can travel directly between vertices of increasing weight without using intermediate edges.
- When $\delta > 2$ and $\gamma < \frac{\delta}{\delta+1}$ for the preferential attachment kernel resp. $\gamma < \frac{1}{2}$ for the product kernel, we expect that the long-range and scale-free effects do not influence the behaviour of the random walk, so that for $d \geq 3$ the infinite cluster would be transient. This is open for the random connection model (and long-range percolation) in general. An analysis of this situation is beyond the scope of the present paper, and postponed to future work.
- We have excluded the boundary cases $\gamma = \frac{1}{2}$, $\gamma = \frac{\delta}{\delta+1}$ and $\delta = 2$ from our consideration, as in these cases the behaviour is typically model-dependent and therefore less suitable for a generalised approach that we develop.

For a summary of the results we refer to Figure 1. We devise a fairly general setup that allows treating more general classes of h, and prove our results for various versions of h. Arguably, the preferential attachment kernel is the most interesting, but also technically the most involved model, and may therefore be considered as the main contribution of the present paper. Indeed, the results for the product kernel are in correspondence with the findings of Heydenreich, Jorritsma and Hulshof [11]. Corresponding results for the plain kernel are due to Sönmez and can be found in [26]. The only known result for the behavior at the critical point β_c concerns the plain kernel (4), see [10]. Note that these papers parametrise the model differently, see Table 2 for relations between them.

TABLE 2. Correspondence of parameters between our paper and [4, 11].

parameters in this paper		parameters in [4, 11]
β	$=$	$\lambda^{d/2}$
δ	$=$	α/d
γ	$=$	$\frac{d}{\alpha(\tau-1)}$

Overview of the paper Before we prove our results, we describe the model in a more rigorous manner in Section 2. Then we prove transience in Section 3, first for the robust case and then for the case $\delta < 2$. Finally, we prove recurrence in dimension 1 and 2 in Section 4.

2. The random connection model with weights

Construction as a point process on $(\mathbb{R}^d \times [0,1])^{[2]} \times [0,1]$. We give now a more formal construction of the random connection model with additional marks. To this end, we extend the construction given in [10, Sections 2.1 and 2.2] by additional vertex marks (the *weight* or birth time). For further constructions, see Last and Ziesche [17] and Meester and Roy [20].

We construct the random connection model as a deterministic functional $\mathcal{G}_\varphi(\xi)$ of a suitable point process ξ. Let η denote a unit intensity $\mathbb{R}^d$-valued Poisson point process, which we can write as

$$\eta = \{X_i \colon i \in \mathbb{N}\}; \tag{9}$$

such enumeration is possible by [16, Corollary 6.5]. In order to define random walks on the random connection model, it is convenient to have a designated (starting) point, and we therefore add an extra point $X_0 = 0$ and thereby get a Palm version of the Poisson point process.

We further equip any Poisson point X_i $(i \geq 0)$ with an independent mark S_i drawn uniformly from the interval $(0,1)$. This defines a point process $\eta' := \{\mathbf{X}_i = (X_i, S_i) \colon i \in \mathbb{N}_0\}$ on $\mathbb{R}^d \times (0,1)$. Let $(\mathbb{R}^d \times (0,1))^{[2]}$ denote the space of all sets $e \subset \mathbb{R}^d \times \mathbb{M}$ with exactly two elements; these are the potential edges of the graph. We further introduce independent random variables $(U_{i,j} : i,j \in \mathbb{N}_0)$ uniformly distributed on the unit interval $(0,1)$ such that the double sequence $(U_{i,j})$ is independent of η'. Using $<$ for the strict lexicographical order on $\mathbb{R}^d$, we can now define

$$\xi := \big\{\big(\{(X_i, S_i), (X_j, S_j)\}, U_{i,j}\big) : X_i < X_j, i, j \in \mathbb{N}_0\big\}, \tag{10}$$

which is a point process on $(\mathbb{R}^d \times (0,1))^{[2]} \times (0,1)$. Mind that η' might be recovered from ξ. Even though the definition of ξ does depend on the ordering of the points of η, its distribution does not. We can now define the weight-dependent random connection model $\mathcal{G}_\varphi(\xi)$ as a deterministic functional of ξ; its vertex and edge sets are given as

$$V(\mathcal{G}_\varphi(\xi)) = \eta', \tag{11}$$

$$E(\mathcal{G}_\varphi(\xi)) = \{\{\mathbf{X}_i, \mathbf{X}_j\} \in V(\mathcal{G}_\varphi(\xi))^{[2]} : X_i < X_j, U_{i,j} \leq \varphi(\mathbf{X}_i, \mathbf{X}_j), i, j \in \mathbb{N}_0\}. \tag{12}$$

Only in this section we write $\mathcal{G}_\varphi(\xi)$ in order to make the dependence on the connection function φ explicit; in the following sections we will fix a kernel function as well as the parameters δ and γ, and therefore only write $\mathcal{G}^\beta = \mathcal{G}^\beta(\xi)$.

Percolation. Our construction ensures that $\mathbf{0} := (X_0, S_0) \in V(\mathcal{G}_\varphi(\xi))$. We now write $\{0 \leftrightarrow \infty\}$ for the event that the random graph $\mathcal{G}_\varphi(\xi)$ contains an infinite self-avoiding path $(v_1, v_2, v_3, \dots)$ of vertices with $v_i \in V(\mathcal{G}_\varphi(\xi))$ $(i \in \mathbb{N})$ such that $\{\mathbf{0}, v_1\}, \{v_1, v_2\}, \{v_2, v_3\} \dots \in E(\mathcal{G}_\varphi(\xi))$, and we say that in this case the graph *percolates*. We denote the percolation probability by

$$\theta(\beta) = \mathbb{P}(0 \leftrightarrow \infty \text{ in } \mathcal{G}_\varphi(\xi));$$

for the probability that this happens; this quantity is well-defined by the monotonicity of the right-hand side in β. This allows us to define the critical percolation threshold as

$$\beta_c := \inf\{\beta > 0 : \theta(\beta) > 0\} \geq 0. \tag{13}$$

Random walks. We recall that, as $\gamma < 1$, the resulting graph $\mathcal{G}_\varphi(\xi)$ is locally finite, i.e.

$$\sum_{\mathbf{y} \in V(\mathcal{G}_\varphi(\xi))} \mathbb{1}\{\{\mathbf{x}, \mathbf{y}\} \in E(\mathcal{G}_\varphi(\xi))\} < \infty \qquad \text{for all } \mathbf{x} \in V(\mathcal{G}_\varphi(\xi)) \text{ almost surely,}$$

cf. [8, p. 8]. Given $\mathcal{G}_\varphi(\xi)$ with $0 \leftrightarrow \infty$ we define the *simple random walk* on the random graph $\mathcal{G}_\varphi(\xi)$ as the discrete-time stochastic process for which $X_0 = 0$ and

$$P^{\mathcal{G}_\varphi(\xi)}(X_n = y \mid X_{n-1} = x) = \frac{\mathbb{1}\big\{\{\mathbf{x},\mathbf{y}\} \in E(\mathcal{G}_\varphi(\xi))\big\}}{\sum_{\mathbf{z}\in V(\mathcal{G}_\varphi(\xi))} \mathbb{1}\big\{\{\mathbf{x},\mathbf{z}\} \in E(\mathcal{G}_\varphi(\xi))\big\}}, \quad \mathbf{x},\mathbf{y} \in V(\mathcal{G}_\varphi(\xi)), n \in \mathbb{N}.$$

We say that $\mathcal{G}_\varphi(\xi)$ is *recurrent* if

$$P^{\mathcal{G}_\varphi(\xi)}\big(\exists n \geq 1 : X_n = 0\big) = 1,$$

otherwise we say that it is *transient*.

3. Transience

In this section we prove the transience results. Throughout, we write $\mathcal{G}^\beta$ instead of $\mathcal{G}_\varphi(\xi)$ to stress that kernel and profile are fixed and the percolation parameter is β.

3.1. **Transience in the robust case.**

Two-connection probability and other properties. We first focus on the case of the preferential attachment kernel (7). As we will see later, this kernel is the most difficult one to consider - the proofs for the sum and min kernel then follow with various simplifications.

We start with the observation, that two sufficiently old vertices are fairly likely to be connected via a younger vertex. This result is a corollary of [13, Lemma A.1].

Lemma 3.1. *Let $\mathbf{x} = (x,s), \mathbf{y} = (y,t)$ be two vertices of $\mathcal{G}^\beta$ with $s,t \leq 1/2$ and define*

$$k(\mathbf{x},\mathbf{y}) = s^{-\gamma}\rho\left(\beta^{-1}t^\gamma\left(s^{-\frac{\gamma}{d}} + |x-y|\right)^d\right)$$

and

$$q(\mathbf{x},\mathbf{y}) = \frac{\rho(1/\beta)\kappa_d}{2}\left(k(\mathbf{x},\mathbf{y}) \vee k(\mathbf{y},\mathbf{x})\right),$$

where κ_d is the volume of the d-dimensional unit ball. Then, with probability at least

$$1 - e^{-q(\mathbf{x},\mathbf{y})},$$

there exists some $\mathbf{z} = (z,u)$ with $u > 1/2$ which is a common neighbour of both $\mathbf{x}$ and $\mathbf{y}$.

Proof. Let X_c denote the set of 'young' common neighbours $\mathbf{z} = (z,u), u > 1/2$ of $\mathbf{x}$ and $\mathbf{y}$, i.e. the vertices which satisfy $U_{\mathbf{x},\mathbf{z}} \leq \rho(\frac{1}{\beta}s^\gamma u^{1-\gamma}|x-z|^d)$ and $U_{\mathbf{y},\mathbf{z}} \leq \rho(\frac{1}{\beta}t^\gamma u^{1-\gamma}|y-z|^d)$. Consider now those vertices $(z,u) \in X_c$ with $|x-z|^d \leq s^{-\gamma}$ and which satisfy $U_{\mathbf{x},\mathbf{z}} \leq \rho(1/\beta)$ and $U_{\mathbf{y},\mathbf{z}} \leq \rho(\frac{1}{\beta}t^\gamma u^{1-\gamma}|t-z|^d)$. Let $X_c^{\mathbf{x}}$ denote the set of those vertices. By the thinning theorem $X_c^{\mathbf{x}}$ forms a Poisson point process with intensity

$$\begin{aligned}\int_{1/2}^1\int_{B_{s^{\gamma/d}}(x)} \rho(1/\beta)\rho(\beta^{-1}t^\gamma u^{1-\gamma}|y-z|^d)dzdu &\geq \frac{\rho(1/\beta)}{2}\int_{B_{s^{\gamma/d}}(x)} \rho(\beta^{-1}t^\gamma|y-z|^d)dz \\ &\geq \frac{\rho(1/\beta)}{2}\int_{B_{s^{\gamma/d}}(x)} \rho(\beta^{-1}t^\gamma(|y-z|+|x-z|)^d)dz \\ &\geq \frac{\rho(1/\beta)\kappa_d}{2}s^{-\gamma}\rho(\beta^{-1}t^\gamma(s^{-\gamma/d}+|x-z|)^d).\end{aligned}$$

Hence,

$$\mathbb{P}(X_c = \emptyset) \leq \mathbb{P}(X_c^{\mathbf{x}} = \emptyset) \leq \exp\left(-\frac{\rho(1/\beta)\kappa_d}{2}k(\mathbf{x},\mathbf{y})\right).$$

Reversing the roles of x and y yields the stated result. □

Remark 3.2. Lemma 3.1 holds precisely as stated also for the min kernel, since all inequalities of the proof hold also in that case.

We will refer to connecting vertices such as in Lemma 3.1 as *connectors*. Next, we show several useful properties of the vertex birth-time distribution that will make it easier to consider the model at different scales. Here and throughout the section we fix the following values

$$u_n = \frac{1}{c_1}(n+2)^{-\frac{k}{\gamma(\delta+1)}} 2^{-\ t\frac{(n+2)d\delta}{\gamma(\delta+1)}} ((n+3)!)^{-\frac{2d\delta}{\gamma(\delta+1)}}, \quad n \in \mathbb{N}, \tag{14}$$

where $k = \frac{2d(\gamma(\delta+1)-\delta)}{\gamma(\delta+1)}$ is a positive constant and

$$c_1 = \left(\tfrac{1}{2}\kappa_d \rho(\tfrac{1}{\beta})\beta^{\delta-1} d^{-d\delta/2}\right)^{-1/\gamma(\delta+1)},$$

where κ_d is the volume of the d-dimensional unit ball. For simplicity, the reader can think of these values as a sequence that decreases towards 0 sufficiently fast.

Lemma 3.3. *Let $(U_i)_{i\leq n}$ be a collection of uniformly on $(0,1)$ i.i.d. random variables. Then, for $0 \leq a < b \leq 1$, it holds that*

$$\mathbb{P}(\min_i U_i > a \,|\, \max_i U_i < b) \leq \exp\{-n\frac{a}{b}\}. \tag{15}$$

Furthermore, for sufficiently large n

$$\mathbb{P}\big(U_i < (\sqrt{d}2^n((n+1)!)^2)^{-d/\gamma} \,|\, U_i < u_{n-2}\big) \leq \exp\{-c_1 n \log(n)\}, \tag{16}$$

where u_{n-2} is as defined above. Finally, for $U, U' \sim$ Unif$(0,1)$ and $x < y$, then

$$\mathbb{P}(U < x \,|\, U < y) = \mathbb{P}(yU' < x). \tag{17}$$

Proof. In order to prove both bounds, we will use the simple fact that $\mathbb{P}(U_i > a \,|\, U < b) = 1 - \frac{a}{b}$. We begin now with the first bound. Since $1 - x \leq e^{-x}$, it holds that

$$\mathbb{P}(\min_i U_i > a \,|\, \max_i U_i < b) = \big(1 - \frac{a}{b}\big)^n \leq \exp\{-n\frac{a}{b}\},$$

which proves the claim. To prove the second bound, note that

$$\frac{\big(\sqrt{d}2^n((n+1)!)^2\big)^{-d/\gamma}}{u_{n-2}} = c_1 2^{-\frac{nd}{\gamma}\frac{1}{\delta+1}} ((n+1)!)^{-\frac{2d}{\gamma}\frac{1}{\delta+1}} n^{\frac{k}{\gamma(\delta+1)}}.$$

Using Stirling's formula with the factorial term and using that the resulting term of the form $e^{-n\log n}$ dominates the term $n^{\frac{k}{\gamma(\delta-1)}}$ for large n yields the result.

The final statement of the lemma follows trivially from

$$\mathbb{P}(U < x \,|\, U < y) = \frac{\mathbb{P}(U < x \cap U < y)}{\mathbb{P}(U < y)} = \frac{x}{y} = \mathbb{P}(yU' < x).$$

□

In order to prove that the age-dependent random connection model exhibits similar behaviour at different scales, we first need to prove the intuitively clear property that the probability of two vertices connecting through a connector vertex as in Lemma 3.1 is, ceteris paribus, decreasing in the distance between the two vertices.

Lemma 3.4. *The probability that a vertex $\mathbf{x}_1 = (x_1, t_1)$ is connected through a connector with $\mathbf{x}_2 = (x_2, t_2)$ is a monotonically decreasing function of the distance $r := |x_2 - x_1|$.*

Proof. We can fix the age of the connector vertex to be $u \in [1/2, 1]$. Furthermore, since our probability space is translation and rotation invariant, we can set $x_1 = \{0\}^d$ and $x_2 = (r, 0, \ldots, 0)$ for $r > 0$. Therefore, the probability that the vertex $\mathbf{x}_1 = (x_1, t_1)$ is connected through a connector of age u to $\mathbf{x}_2 = (x_2, t_2)$ is a constant multiple of

$$\int_{\mathbb{R}^d} \rho(c_1|x|^d)\rho(c_2|x_2 - x|^d)dx$$

where we have written all factor that do not depend on x and x_2 as constants c_1 and c_2.

We first prove the claim for $d = 1$. In this case, the problem simplifies to showing that

$$\int_{\mathbb{R}} \rho(c_1|x|)\rho(c_2|r-x|)dx$$

is a non-increasing function of r. Since ρ is integrable, bounded and monotone, we have that

$$\begin{aligned}\frac{d}{dr}\int_{\mathbb{R}} \rho(c_1|x|)\rho(c_2|r-x|)dx &= \int_{\mathbb{R}} \rho(c_1|x|)\frac{d}{dr}\rho(c_2|r-x|)dx \\ &= \int_{\mathbb{R}} \rho(c_1|x|)\rho'(c_2|r-x|)c_2 \operatorname{sign}(r-x)dx\end{aligned}$$

We now separate the integral into two parts

$$-\int_{x>r} \rho(c_1|x|)\rho'(c_2|r-x|)c_2 dx \quad \text{and} \quad \int_{x\le r} \rho(c_1|x|)\rho'(c_2|r-x|)c_2 dx$$

and observe that since $\rho'(c_2|r-x|)$ is symmetric around r, it suffices to compare the behaviour of $\rho(c_1|x|)$ as $|x-r|$ increases. Since ρ is non-increasing, this gives that the right term is bigger in absolute value at every x, and since $\rho'(c_2|r-x|)$ is non-positive for all x, this yields that $\frac{d}{dr}\int_{\mathbb{R}} \rho(c_1|x|)\rho(c_2|r-x|)dx$ is non-positive for all $r > 0$, which proves the claim.

We now proceed to prove the claim for $d \ge 2$. We again differentiate with respect to r to obtain

$$\frac{d}{dr}\int_{\mathbb{R}^d} \rho(c_1|x|^d)\rho(c_2|x_2-x|^d)dx = \int_{\mathbb{R}^d} \rho(c_1|x|^d)\rho'(|x_2-x|^d)|x_2-x|^{d-1}\frac{(x_2-x)}{|x_2-x|}dx,$$

where $\frac{(x_1-x)}{|x_1-x|}$ is a vector of unit length. Recalling that $x_2 = (r, 0, \cdots, 0)$ and writing $\mathbf{e}_i$ for the i-th basis vector and $\langle \cdot, \cdot \rangle$ for the inner product, we have that

$$\left\langle \frac{d}{dr}\int_{\mathbb{R}^d} \rho(c_1|x|^d)\rho(c_2|x_2-x|^d)dx, \mathbf{e}_i \right\rangle$$

is 0 for all $i \neq 1$ and equal to

$$\int_{\mathbb{R}} \rho(c_1|x|^d)\rho'(|x_2-x|^d)|x_2-x|^{d-1} \operatorname{sign}(r-x_2)dx_2$$

for $i = 1$. Using the same argument as in the case $d = 1$ yields that this expression is non-positive. Therefore, since any vector z for which $|x_2 + \epsilon z| > |x_2| \forall \epsilon > 0$ has $z_1 \ge 0$, the directional derivative ∇_z of $\int_{\mathbb{R}^d} \rho(c_1|x|^d)\rho(c_2|x_2-x|^d)dx$ is (due to the linearity of the inner product) also non-positive, which concludes the proof. $\square$

Remark 3.5. Note that Lemma 3.4 depends only on the profile function ρ and not on the choice of the kernel function and therefore holds for all models considered.

The following lemma will show a very useful relationship between the model with parameter β and a coarse-grained version of the model with an appropriately modified parameter β. With that in mind, we will use $\mathbb{P}_\beta$ to denote the law with a given edge density parameter β.

Lemma 3.6. *Let $N \in \mathbb{N}$ and let $Q_i := Nv_i + [0, N]^d$ for $i \in \{1, 2\}$, where $v_1, v_2 \in \mathbb{Z}^d$ and $|v_1 - v_2| = k$ for some $k \in \mathbb{R}_+$. Furthermore, let $\mathbf{x}_i = (x_i, u_i) := \arg\min_{(x,t)\in Q_i} t$ for $i \in \{1, 2\}$. Then, for $\lambda > 0$ it holds that*

$$\begin{aligned}&\mathbb{P}_\beta\left(\substack{\mathbf{x}_1 \text{ is connected through} \\ \text{a connector to } \mathbf{x}_2} \,\middle|\, u_1, u_2 \le \lambda^{-1/\gamma} N^{\frac{d(1-2\delta)}{2\delta\gamma}} \right) \ge \\ &\qquad \mathbb{P}_{\beta^*}\left(\substack{(v_1,t_1) \text{ is connected through} \\ \text{a connector to } (v_2,t_2)} \,\middle|\, (v_1, t_1) \text{ and } (v_2, t_2) \text{ are occupied} \right),\end{aligned} \tag{18}$$

where $\beta^* = \beta\lambda d^{\frac{d(1-2\delta)}{4\delta}}$ and t_1, t_2 are two independent, uniformly distributed random variables.

Proof. We start by considering the left hand side of (18) and write $b = \lambda N^{\frac{d(1-2\delta)}{2\delta\gamma}}$ to simplify notation. Since $|x_2 - x_1| \le \sqrt{d}|Nv_2 - Nv_1|$ we have due to (17) and Lemma 3.4 that

$$\int_{\mathbb{R}^d}\int_{1/2}^{1} \mathbb{P}(\mathbf{x}_1 \text{ is connected through } (x,u) \text{ to } \mathbf{x}_2 \,|\, u_1, u_2 \le b) du dx$$
$$\ge \int_{\mathbb{R}^d}\int_{1/2}^{1} \mathbb{P}((\sqrt{d}Nv_1, bt_1') \text{ is connected through } (x,u) \text{ to } (\sqrt{d}Nv_2, bt_2')) du dx,$$

where t_1' and t_2' are independent uniform random variables. Since all edges exist independently of each other, we can write this as

$$\int_{\mathbb{R}^d}\int_{1/2}^{1} \prod_{i=1}^{2} \rho\Big(\frac{1}{\beta} u^{1-\gamma}(bt_i')^{\gamma}|\sqrt{d}Nv_i - x|^d\Big) du dx$$
$$\ge \int_{\mathbb{R}^d}\int_{1/2}^{1} \prod_{i=1}^{2} \rho\Big(\frac{1}{\beta}((\sqrt{d}N)^{-\frac{d}{2\delta}} b^{\gamma} u^{1-\gamma}(t_i')^{\gamma}|\sqrt{d}Nv_i - \sqrt{d}Ny|^d\Big) du dy,$$

where we have used a change of variable and that $\rho(x) = 1 \wedge x^{-\delta}$ and therefore non-increasing. Setting

$$\beta^* := \beta(\sqrt{d}N)^{\frac{d(1-2\delta)}{2\delta}} b^{-\gamma}$$

and writing the ρ terms again as edge probabilities yields the desired result. $\square$

Remark 3.7. The exponent of N in (18) is specific to the preferential attachment kernel. The lemma holds for other kernels when the exponent is appropriately modified.

Proof of transience. We now have all the necessary tools in place to prove that when $\gamma > \frac{\delta}{1+\delta}$ (i.e. we are in the robust case), then the infinite component of the weight-dependent random connection is transient. We prove the results in two stages - first, we show that the statement holds when the edge density parameter β is sufficiently large. We do this by showing the existence of a renormalised graph sequence, which by [22, 1] implies that the underlying graph is transient. Next, we use a coarse graining argument to show that at a sufficiently large scale, one can observe transience even at smaller values of β. In order to do the latter, we will need to introduce an additional thinning parameter. Let $p \in (0, 1]$ be the *retention probability* with which we retain each vertex of the graph, independently from all other vertices and everything else. Equivalently, with probability $1 - p$ a vertex (and therefore all its neighbouring edges) are removed from the graph. Then, in Proposition 3.8 below we consider *bond-edge percolation*, i.e. the graph obtained by first removing each vertex of the graph with probability $1 - p$ and then connecting the remaining vertices according to φ as before.

Proposition 3.8. *Consider the age-dependent random connection model with $\gamma > \frac{\delta}{1+\delta}$, i.e. the robust case. Then, for any $p \in (0, 1]$ and β large enough, the infinite percolation cluster is transient.*

Proof. First observe that when $\gamma > \frac{\delta}{1+\delta}$, there exists a unique infinite percolation cluster which is robust. Therefore w.l.o.g. we set $p = 1$ and note that up to different constants, the proof goes through unchanged for all $p > 0$. Define for $n \in \mathbb{N}$ the values

$$C_n := (n+1)^{2d}, \quad D_n := 2(n+1)^2,$$

and recall that the value of u_n and k from (14). Note that $\gamma > \frac{\delta}{1+\delta}$ implies that $k > 0$. We partition $\mathbb{R}^d$ into disjoint boxes of side length D_1; we call them 1-stage boxes and we furthermore call the vertices of the graph 0-stage boxes. We now define the renormalization procedure. We partition $\mathbb{R}^d$ again, grouping D_2^d 1-stage boxes together to form 2-stage boxes. We continue like this for all $n \ge 3$, so that the n-stage boxes represent a partitioning of $\mathbb{R}^d$ into boxes of side length $\prod_{i=1}^{n} D_i$.

We next define what it means for a box to be "good" or "bad", starting with 0-stage boxes. Here, we declare all vertices of the (thinned) Poisson point process as good. We declare a 1-stage box as good, if it contains at least C_1 good vertices and at least one of these vertices has birth-time at most u_1; for each 1-stage box we declare the oldest vertex (i.e., the vertex with the smallest birth-time) with birth-time smaller than u_1 to be 1-dominant. For $n \geq 2$, we declare a n-stage box Q good if the following three conditions hold:

(E): At least C_n of the $(n-1)$-stage boxes in Q are good.

(F): For any two (not necessarily different) $(n-1)$-stage boxes $Q', Q'' \subset Q$, every pair of $(n-2)$-dominant vertices in Q' and Q'' is connected.

(G): There is an $(n-1)$-dominant vertex in one of Q's good $(n-1)$-stage boxes, with birth-time at most u_n.

We declare for each good n-stage box the vertex with the smallest birth-time as the n-dominant vertex, given that its birth-time is smaller than u_n. We now define $E_n(v)$, $F_n(v)$ and $G_n(v)$ to be the events that conditions (E), (F) and (G) hold for the n-stage box containing vertex v. When considering the origin, we omit the vertex in this notation. We do the same for the event $L_n(v)$, which we define to be the event that the corresponding n-stage box is good.

Due to translation invariance it is enough to show that

$$\mathbb{P}\Big(\bigcap_{n=1}^{\infty} L_n\Big) > 0$$

and since the events L_n are positively correlated, this can be further simplified to showing that

$$\prod_{n=1}^{\infty} \mathbb{P}(L_n) > 0.$$

We now first upper bound the probability of the converse event L_n^c for large n and begin by writing

$$\mathbb{P}(L_n^c) \leq \mathbb{P}(E_n^c) + \mathbb{P}(F_n^c \,|\, E_n) + \mathbb{P}(G_n^c \,|\, E_n). \tag{19}$$

To bound $\mathbb{P}(F_n^c \,|\, E_n)$, note that any two vertices in the same n-stage box are at most

$$\sqrt{d} \prod_{k=1}^{n} D_k = \sqrt{d} 2^n ((n+1)!)^2$$

away from each other. Let A_n be the event that two $(n-2)$-dominant vertices (x, s) and (y, t) in the same n-stage box are not connected via a connector vertex. Then, using Lemmas 3.1 and 3.3 we obtain that

$$\begin{aligned}
\mathbb{P}(A_n) &\leq \mathbb{P}\big(A_n \,|\, s, t \geq (\sqrt{d} 2^n ((n+1)!)^2)^{-d/\gamma}\big) + \mathbb{P}\big(s, t < (\sqrt{d} 2^n ((n+1)!)^2)^{-d/\gamma}\big) \\
&\leq \exp\Big\{ -\frac{1}{2} \rho\Big(\frac{1}{\beta}\Big) \kappa_d \beta^\delta u_{n-2}^{-\gamma(\delta+1)} d^{-\frac{d\delta}{2}} 2^{-nd\delta} ((n+1)!)^{-2d\delta} \Big\} + \exp\{-c_1 n \log(n)\} \\
&\leq \exp\{-\beta n^k\} \vee \exp\{-c_1 n \log(n)\}.
\end{aligned}$$

There are

$$\binom{D_n^d D_{n-1}^d}{2} < 4^d (n+1)^{4d}$$

possible pairs of $n-2$-stage boxes in an n-stage box and therefore at most $4^d(n+1)^{4d}$ connections via a connector vertex. By taking the union bound, we obtain

$$\mathbb{P}(F_n^c \,|\, E_n) \leq \exp\{d \log(4) + 4d \log(n+1) - \beta n^k \wedge c_1 n \log(n)\}. \tag{20}$$

We next bound $\mathbb{P}(G_n^c \,|\, E_n)$. For some positive constant c_2 we can write

$$\begin{aligned}\frac{u_n}{u_{n-1}} &= \Big(\frac{n}{n-1}\Big)^{-\frac{k}{\gamma(\delta+1)}} 2^{-\frac{d\delta}{\gamma(\delta+1)}} (n+1)^{-\frac{2d\delta}{\gamma(\delta+1)}} \\ &\geq c_2(n+1)^{-\frac{2d\delta}{\gamma(\delta+1)}}\end{aligned}$$

and therefore using Lemma 3.3 we obtain

$$\begin{aligned}\mathbb{P}(G_n^c \,|\, E_n) &\leq \exp\Big\{-C_n\frac{u_n}{u_{n-1}}\Big\} \\ &\leq \exp\Big\{-c_2(n+1)^{\frac{2d(\gamma(\delta+1)-\delta)}{\gamma(\delta+1)}}\Big\}. \qquad (21)\end{aligned}$$

Note that the exponent of the term $n+1$ is precisely the value k and therefore positive.

We now proceed to bound the remaining term of (19) by using Chernoff's bound that says that if $X \sim \text{Bin}(m,p), \Theta \in (0,1)$, then $\mathbb{P}(X < (1-\Theta)mp) \leq \exp\{-\frac{1}{2}\Theta^2 mp\}$. This leads to

$$\begin{aligned}\mathbb{P}(E_n^c) &\leq \exp\Big\{-\frac{1}{2}\Big(1-\frac{1}{2^d}\frac{1}{\mathbb{P}(L_{n-1})}\Big)^2\mathbb{P}(L_{n-1})(n+1)^{2d}\Big\}, \\ &= \exp\{-2^{-d-1}(2^d\mathbb{P}(L_{n-1}-1)^2(n+1)^{2d}\} \qquad (22)\end{aligned}$$

where we used the definitions of C_n, D_n and the definition of E_n itself. Combining (22), (20) and (21) into (19), we obtain the recursive inequality

$$\begin{aligned}\mathbb{P}(L_n^c) \leq{}& \exp\{d\log(4) + 4d\log(n+1) - \beta n^k \wedge c_1 n\log(n)\} \\ &+ \exp\{-c_2 n^k\} + \exp\{-2^{-d-1}(2^d\mathbb{P}(L_{n-1})-1)^2(n+1)^{2d}\}.\end{aligned}$$

For β large enough there exists n_0 such that for all $n > n_0$ it holds

$$\mathbb{P}(L_n^c) \leq 2\exp\{-c_3 n^k \wedge n\log(n)\} + \exp\{-2^{-d-1}(2^d\mathbb{P}(L_{n-1})-1)^2(n+1)^{2d}.\}$$

Define now the sequence $\ell_n := 1-(n+1)^{-3/2}$ and observe that $\prod_{i=1}^{\infty}\ell_i > 0$. Next, note that for any $n_1 > n_0$ we can find a β_0 large enough, such that $\mathbb{P}(L_{n_1}) \geq \ell_n$, which holds since the event L_{n_1} depends only on the vertices and edges inside a finite box and can be made arbitrarily likely by increasing β (see the remarks for (2.1) in [8] for details about the sufficiency of this). We now write

$$\begin{aligned}\mathbb{P}(L_n^c) &\leq 2\exp\{-c_3 n^k \wedge n\log(n)\} + \exp\{-2^{-d-1}(2^d(1-2^{-3/2})-1)^2(n+1)^{2d}\} \\ &\leq (n+1)^{-3/2} \\ &= 1-\ell_n,\end{aligned}$$

where we chose n_1 so large that the last inequality holds. We can now write

$$\prod_{n=1}^{\infty}\mathbb{P}(L_n) = \prod_{n=1}^{n_1}\mathbb{P}(L_n)\prod_{n=n_1+1}^{\infty}\mathbb{P}(L_n) \geq \prod_{n=1}^{n_1}\mathbb{P}(L_n)\prod_{n=n_1+1}^{\infty}\ell_n > 0.$$

This gives the existence of the renormalized graph sequence with positive probability, and by the multi-scale ansatz of [22, 1] the result follows. ☐

In order to prove that in the robust regime, the infinite cluster is transient for *all* $\beta > \beta_c$, we first need to extend the result of Proposition 3.8 to a discretized version of the problem. More precisely, let φ remain defined as it was, but instead of working with a Poisson point process on $\mathbb{R}^d$, consider instead the open sites of independent site percolation on $\mathbb{Z}^d$ with retention parameter p. We claim that Proposition 3.8 holds also on this discrete version of the model.

Corollary 3.9. *Consider the age-dependent random connection model with $\gamma > \frac{\delta}{1+\delta}$, on the lattice site percolation cluster with retention probability $p > 0$. Then, for β large enough, the infinite percolation cluster is transient.*

Proof. It is easy to see that the proofs of Lemmas 3.1, 3.4 and 3.6 go through by replacing the relevant spatial integrals with sums across lattice vertices and therefore the claims of the lemmas remain (up to constants that depend only on d) unchanged.

Therefore, any applications of these lemmas in the proof of Proposition 3.8 stay as they are and we only need to explain how the initial definition of the conditions (E), (F) and (G) changes. In fact, the only change necessary is to the definition of good 0-stage boxes. Before, we defined all vertices of the thinned Poisson point process to be good 0-stage boxes. Analogously we now define a site of $\mathbb{Z}^d$ to be a good 0-stage box if it is *open* and bad if it is *closed* (which happens with probability p and $1-p$ respectively). The remaining definitions remain as they are and due to the discrete nature of the tessellation into boxes, the remainder of the proof of Proposition 3.8 goes through without further changes. $\square$

Proposition 3.10. *Consider the age-dependent random connection model with* $\gamma > \frac{\delta}{1+\delta}$ *and* $\beta > \beta_c$. *Then, the infinite percolation cluster is transient.*

Proof. For the case when β is sufficiently large, the result holds by Proposition 3.8. It remains to show that the result holds for all $\beta > 0$.

We partition $\mathbb{R}^d$ into cubes of side length N, for arbitrary fixed $N > 0$. In every cube we identify the vertex with the smallest birth-time and call it the dominant vertex. We now choose λ large enough, so that $\beta^* = \beta\lambda d^{d(1-2\delta)/(4\delta)}$ is sufficiently large for Proposition 3.8 to hold. Next, let E_N be the event that a dominant vertex (if it exists) has birth-time smaller than $\lambda^{-1/\gamma}N^{d(1-2\delta)/(2\delta\gamma)}$ and let $p_N := \mathbb{P}(E_N)$. Furthermore, we declare every cube on which this event occurs as good. Observe that although p_N is decreasing in N, it is strictly positive for every finite N. Therefore by Lemma 3.6, the status of the connections via a connector vertex between the dominant vertices in good cubes for the model with parameter β stochastically dominates the status of the connections for a discrete version of the model with parameter β^* and thinning probability p_N. By Corollary 3.9 the infinite component for this set of parameters is transient, which establishes the claim. $\square$

This concludes the proof of Theorem 1.1 for the preferential attachment kernel and $\gamma > \frac{\delta}{\delta+1}$. The kernels $h^{\min}$ and h^{sum} lead to the same transient behaviour as h^{pa}, so it should come as no surprise that the result follow by a similar proof. Moreover, by observing that

$$h^{\min}(s,t,v) \leq h^{\text{sum}}(s,t,v) \leq \frac{1}{2}h^{\min}(s,t,v),$$

it suffices to show the claim for $h^{\min}$.

Proposition 3.11. *Consider the weight-dependent random connection model with min or sum kernel with* $\gamma > \frac{\delta}{\delta+1}$ *and* $\beta > \beta_c$. *Then, the infinite percolation cluster is transient.*

Proof. We need to use the same two-connection strategy as for the preferential attachment kernel. In fact, by using the min kernel $h^{\min}$ instead of h^{pa}, one obtains precisely the same bound in Lemma 3.1 (see Remark 3.2). Likewise, by Lemma 3.4 $h^{\min}$ is monotonically decreasing in the distance (see Remark 3.5) and therefore up to a change of exponents in Lemma 3.6 (see Remark 3.7), the model can be similarly rescaled. Finally, Proposition 3.8, Corollary 3.9 and Proposition 3.10 then follow under the same conditions that $\gamma > \frac{\delta}{\delta+1}$. $\square$

This concludes the proofs for the kernels that behave similarly to h^{pa}, so we now focus on the weight-dependent random connection model with the remaining kernels of Theorem 1.1. Note that the result for the product kernel has been shown for a lattice based version of the model in [11], so we only highlight the changes needed to prove the result in our framework.

Proposition 3.12. *Consider the weight-dependent random connection model with product kernel with* $\gamma > \frac{1}{2}$ *and* $\beta > \beta_c$. *Then, the infinite percolation cluster is transient.*

Proof. The result follows from [11], noting that all steps of the proofs stay as they were, with the exception of the initial step of the recursion in the proof of [11, Proposition 5.3], which is lattice based. Therefore, we only need to modify the definition of good 0-stage boxes. As in our proof of Proposition 3.8, we define the points of the (thinned) Poisson point process to all be good 0-stage boxes and proceed through the rest of the proof of [11, Proposition 5.3] without any further alterations. Then, our claim follows by applying this modified version of Proposition 5.3 from [11] to prove the result for a sufficiently large value of $\beta > \beta_c$ (analogous to Proposition 3.8 in this paper) and then making the same coarse graining argument as in [11] by using the unmodified version of Proposition 5.3 to prove the claim for general $\beta > \beta_c$ (analogous to Proposition 3.10). $\square$

The last kernel left to consider is the max kernel $h^{\max}$. Here, just like in the product kernel case, direct connections between vertices suffice to show that the graph is transient.

Proposition 3.13. *Consider the weight-dependent random connection model with max kernel with $\gamma > 0$. Then, the infinite percolation cluster is transient.*

Proof. Since it is possible to obtain the desired transience result already with direct connections, Lemmas 3.1 and 3.4 do not need to be applied. The result of Lemma 3.6 holds (with a change of exponent) also for the max kernel $h^{\max}$ when instead of using a connector to connect two vertices, one uses direct connections instead. Similarly, Proposition 3.8 (and Corollary 3.9) also holds by following the exact same steps, using the direct connection probabilities instead of the two-connection bound from Lemma 3.1. By setting

$$u_n := \frac{1}{c_1}(n+2)^{-\frac{k}{\delta(1+\gamma)}} 2^{-\frac{(n+2)d}{1+\gamma}} ((n+3)!)^{-\frac{2d}{1+\gamma}}$$

where c_1 is the same constant as before and $k = \frac{\gamma}{1+\gamma}$, we obtain the desired bounds on $\mathbb{P}(F_n^c \,|\, E_n)$ and $\mathbb{P}(G_n^c \,|\, E_n)$ (see equations (20) and (21)). Furthermore, note that k is strictly positive for all $\gamma > 0$ and therefore the claim of the theorem holds for all $\gamma > 0$ as soon as we observe that the rest of the proof of Proposition 3.8 goes through with no further changes necessary. $\square$

3.2. **Transience in the non-robust case.**

The transience result. This section is primarily devoted to the non-robust age-dependent random connection model, i.e $\rho(v) \sim cv^{-\delta}$ and $h = h^{\mathrm{pa}}$, see (7), with $\delta \in (1,2)$ and $\gamma \leq \delta/(\delta+1)$, $\beta > \beta_c$. However, it will become clear below that in the non-robust regime the precise form of h is not very important.

On the one hand, the non-robust regime is rather delicate, since one does not have the backbone of dominant vertices present that were used to construct the embedded transient graph in the robust case. On the other hand, the assumption of polynomial decay of connection probabilities in terms of distance allows us to employ a variation of Berger's renormalisation argument in [1] for transience of long-range percolation clusters. A similar method was used by Deprez et. al. in [5, 6], the model therein essentially corresponds to our product kernel. We give an enhancement of the proof which allows us to tackle general kernels.

The basic idea is rather straightforward: Under the assumption of polynomial decay of the connection probability with a sufficiently small exponent and a sufficiently large density of vertices whose weight exceeds a given bound, we essentially construct a supercritical long-range percolation cluster of such vertices, which is known to be transient.

Formally, our assumptions is that there exists $s_* \in (0,1]$ such that

$$\liminf_{v\to\infty} \rho(h(s_*, s_*, v))v^{\delta d} > 0. \tag{23}$$

The general transience result in the non-robust situation is given as follows:

Proposition 3.14. *Let $\mathcal{G}^\beta$ denote the weight-dependent random connection model and let h satisfy* (23)*. Assume that $\beta > \beta_c$, then the infinite cluster is transient.*

Let B denote a non-empty, convex subset of $\mathbb{R}^d$, a *cluster in B* is a connected subgraph of $\mathcal{G}$ with all vertex positions in B. The key result of this section is the following statement concerning percolation on finite boxes.

Proposition 3.15 (Local density of percolation clusters). *Let $\beta > \beta_c$. For any $\lambda \in (0,1)$, and any $\varepsilon > 0$, there is a sufficiently large $M_0 \in \mathbb{N}$, such that with probability exceeding $1-\varepsilon$, the box $[0,M)^d$, $M > M_0$, contains a cluster with at least $M^{\lambda d}$ vertices.*

We postpone the proof of Proposition 3.15 and first show how to obtain Proposition 3.14.

Proof of Proposition 3.14. We apply a coarse-graining argument and relate the continuum model to a lattice model. To this end, we consider a tessellation of $\mathbb{R}^d$ by copies of the box $[0,M)^d$. Fix $\varepsilon \in (0,1)$ and $\lambda \in (\delta/2, 1)$, their explicit choice being specified below. A box is *good* if it contains a cluster C_B with at least $M^{\lambda d}$ vertices, if there is more than one such cluster pick one arbitrarily. By Proposition 3.15, we may choose M so large that boxes are good with probability exceeding $1-\varepsilon$. Furthermore, we say that a vertex $\mathbf{x} = (x,s)$ is *old*, if $s < s_*$. Let η_o denote the Poisson point set of old vertices. We say that two boxes B_1, B_2 are k boxes away from one another, if their origins are at graph distance k in the rescaled integer lattice $M\mathbb{Z}^d$. For good boxes B_1, B_2, which are k boxes away from each other, let us calculate a lower bound for the probability that the clusters C_{B_1}, C_{B_2} are connected in $\mathcal{G}$. It holds that

$$\mathbb{P}(C_{B_1} \leftrightarrow C_{B_2}) \geq \mathbb{P}\big(\exists (x_1,s_1),(x_2,s_2) \in \eta_o :\, x_1 \in C_{B_1}, x_2 \in C_{B_2}, ((x_1,s_1),(x_2,s_2)) \in E(\mathcal{G})\big).$$

We claim that $|\eta_o \cap C_B|$ dominates an Binomial($|C_B|, s_*$) r.v., which we verify in Lemma A.8. Since the potential connections between the clusters are independent of the connections within the respective boxes, we can thus bound $|\eta_o \cap C_B|$ below by independent Bin($|C_B|, s_*$) variables. It follows that we may choose a small $c \in (0,1)$, such that, for sufficiently large clusters C_{B_1}, C_{B_2},

$$\mathbb{P}(C_{B_1} \not\leftrightarrow C_{B_2}) \leq \mathrm{e}^{-cs_*^2|C_{B_1}||C_{B_2}|\rho(\frac{1}{\beta}h(s_*,s_*,k\sqrt{d}M))} + 2\mathrm{e}^{-c'\min(|C_{B_1}|,|C_{B_2}|)}$$

where c' is some constant and the term on the right hand side accounts for the possibility that either Binomial is small, using e.g. Bernstein's inequality. Recalling that the boxes B_1, B_2 are assumed to be good and applying (23), we obtain

$$\mathbb{P}(C_{B_1} \leftrightarrow C_{B_2}) \geq 1 - \mathrm{e}^{-\frac{\beta^{\delta d} C s_*^2 M^{(2\lambda-\delta)d}}{k^{\delta d}}} - 2\mathrm{e}^{-c' M^{\lambda d}}, \tag{24}$$

for some suitably chosen constant $C < \infty$ and note that the third term may be absorbed into the constant C, since $\lambda < \delta$. The events $\mathbb{P}(C_{B_i} \leftrightarrow C_{B_j})$ are conditionally independent for different pairs of boxes and boxes are good independently of each other. Hence the renormalised lattice model is transient for sufficiently small ε and any $\beta > \beta_c$, by the transience result for site-percolated long-range percolation [1, Lemma 2.7], since the factor $M^{(2\lambda-\delta)d}$ in the exponential in (24) can be made arbitrarily large. □

Remark 3.16. (i) In fact, for $k = 1$, the estimate (24) yields that the coarse-grained random connection model dominates a (vertex-percolated) supercritical nearest neighbour bond percolation cluster, which is sufficient for transience if $d \geq 3$.

(ii) Given Proposition 3.15, it is straightforward to extend further results from [1] to the model $\mathcal{G}_\varphi(\xi)$. E.g. that there is no infinite cluster at criticality, and that the giant cluster has positive density in large boxes with extremely high probability, c.f. the proofs of Theorems 3.3 and 3.4 as well as Corollary 3.5 in [6].

Local density of percolation clusters. We now turn to the proof of Proposition 3.15. Throughout this section we use the following notation: $B_n := [0,n)^d, n \geq 1$ for the cube of side length n with origin 0 and $B_n^k := \{x : \exists y \in B_n \text{ with } |x-y|_\infty \leq k\}$ its k-neighbourhood (with respect to the ℓ^∞-metric); C_∞ denotes the infinite cluster in $\mathcal{G}$. Let $X \subset Y \subset \eta$. We let $\mathcal{G}_X := (X, \{(\mathbf{x},\mathbf{y}) \in E(\mathcal{G}) : x,y \in X\})$ denote the subgraph of $\mathcal{G}$ induced by X and we say that X *is connected in* Y, if X is connected in $\mathcal{G}_Y$. For $B \subset \mathbb{R}^d$, we will often abuse notation and write just B to refer to the vertices in B instead of $\{\mathbf{x} = (x,s) \in \eta' \colon x \in B\}$. Recall that, formally, the edge weights in (10) are indexed according to the lexicographical order of the incident vertices. In the following, we will use the more practical labelling

$$U_{\mathbf{x},\mathbf{y}}, \quad \mathbf{x},\mathbf{y} \in \eta',$$

to denote weights assigned to specific edges.

We start with an observation on the density of old vertices in the infinite cluster. Let $s_* \in (0,1]$ be such that (23) is satisfied, and

$$C'_n = \{(x,s) \in C_\infty : s \le s_*, x \in B_n\}.$$

Lemma 3.17. *Fix $\beta > \beta_c$, $0 < \theta' < \theta(\beta)$, and $\varepsilon > 0$. There exists N_0, s.t. for all $n \ge N_0$ there is some $k = k(n) > 0$, such that*

$$\mathbb{P}(|C'_n| \ge s_*\theta' n^d, C'_n \text{ is connected in } B_n^k) \ge 1 - \varepsilon.$$

Proof. We have

$$\mathbb{P}(|C'_n| \ge s_*\theta' n^d) = \mathbb{P}\left(\sum_{\mathbf{x}\in B_n} \mathbb{1}\{\mathbf{x} = (x,s) \in C_\infty, s \le s_*\} \ge s_*\theta' n^d\right).$$

By ergodicity and standard results on marked point processes (see e.g. [3, Chapter 13.4]) it holds true that, in probability,

$$s_*^{-1} n^{-d} \sum_{(x,s)\in B_n : s \le s_*} \mathbb{1}\{(x,s) \in C_\infty\} \overset{n\to\infty}{\longrightarrow} \theta(\beta).$$

It follows that we can find n_1, such that for all $n \ge n_1$, we have

$$\mathbb{P}(n^{-d}|C'_n| \ge s_*\theta') \ge 1 - \varepsilon/2.$$

Fix such n and observe that by uniqueness of the infinite cluster, we must have that C'_n is part of the same cluster within B_n^K for some random $K < \infty$. Thus follows the existence of $k(n)$, such that

$$\mathbb{P}(C'_n \text{ is not connected in } B_n^{k(n)}) < \varepsilon/2,$$

and the statement follows by taking a union bound. □

We now define a hierarchical renormalisation scheme, using boxes of the type considered in Lemma 3.17 as basic building blocks. We use the notations $B_n(x) = B_n + x$ and $B_n^k(x) = B_n^k + x$. Our renormalisation is parametrised by a *scaling-sequence* $\sigma = (\sigma_1, \sigma_2, \dots)$ with $\sigma_i \in \{2, 3, \dots\}$, $i \in \mathbb{N}$. σ contains the contraction factors used to renormalise, i.e. σ_1 tells us how many level-0-boxes are contained in a level-1 box and so on. We also require a sequence $\theta = (\theta_0, \theta_1, \dots)$ of *density parameters* with $\theta_0 \in (0, \theta(\beta))$ and $\theta_i \in (0,1)$, $i \in \mathbb{N}$.

We now say that $\{B_{m_l}(x), x \in m_l\mathbb{Z}^d\}$ are the *level-l*-boxes, $l = 1, 2, \dots,$ where $m_l = m_0 \prod_{j=1}^l \sigma_j$ and $m_0 > N_0$ with N_0 such that the conclusion of Lemma 3.17 holds. Note that each box B at level l contains precisely σ_l^d smaller boxes at level $l-1$, which we call the *sub-boxes* of B.

Henceforth, B will always denote some box in our renormalisation scheme. Fix $k \ge k(m_0)$ as in Lemma 3.17. We say that B contains a (k, L)-precluster, if there are at least L vertices of weight s_* in B and they are all part of the same connected component in $\mathcal{G}_{|B^k}$. For ease of notation set $\sigma_0 = m_0$. The following recursive definitions apply:

Level 0 A 0-level box B is *healthy*, if it contains a $(k, \theta_0\sigma_0^d)$-precluster.

Level l Let $l \ge 1$. An l-level box B is *alive*, if it contains at least $\theta_l\sigma_l^d$ healthy $(l-1)$-level boxes. It is *healthy* if all $(k, \prod_{j=0}^{l-1}\theta_j\sigma_j^d)$-preclusters in all healthy (($l-1$)-level boxes contained in B are part of the same cluster of $\mathcal{G}_{B^k}$.

Let $a_l, l \ge 1$, denote the probability that an l-level box is alive and $h_l, l \ge 0$ the probability that it is healthy. We have the following elementary recursion:

Lemma 3.18 ([6, Lemma 6.3])**.** *For all $l \ge 1$,*

$$1 - a_l \le \frac{1 - h_{l-1}}{1 - \theta_l}.$$

Obtaining bounds on $h_l, l = 1, 2, \dots$ is more involved. To state the bounds, we need to carefully specify the parameters of the renormalisation scheme. Recall that $\delta \in (1, 2)$ and fix $\delta' \in (\delta, 2)$. Now chose $\theta_i, i \geq 1$ and $\sigma_i, i \geq 0$ such that

$$\left\lceil \Big(d^{d/2} \prod_{j=0}^{l} \sigma_j^d \Big)^{\delta/\delta'} \right\rceil < \left\lfloor s_* \prod_{j=0}^{l-1} \theta_j \sigma_j^d \right\rfloor, \quad l \geq 1. \tag{25}$$

Lemma 3.21 below provides a specific choice for σ and θ satisfying (25). We can now state the central estimate on the probabilities $h_l, l \geq 1$.

Lemma 3.19. *Let $\delta' \in (\delta, 2)$ and let $\sigma = (\sigma_i)_{i=0}^{\infty}$ and $\theta = (\theta_i)_{i=0}^{\infty}$ satisfy (25), then there exist $\varrho \in (0, 1)$ satisfying*

$$18\varrho > 16 + \delta' \tag{26}$$

and $\zeta(\delta', \varrho)$, such that

$$1 - h_l \leq \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right)^{-\zeta} + 4\mathrm{e}^{-(1-2/\mathrm{e}) s_* \prod_{i=0}^{l-1} \sigma_i^d}.$$

To prove Lemma 3.19 we need a result on a class of inhomogeneous random graphs. To state it, we let $\delta_0 \in (1, 2)$, $\mathbf{m} = (m_1, \dots, m_r)$ with $r \in \mathbb{N}$ and $m_j \in \mathbb{N}, j = 1, \dots, r$ and let $\mathcal{I}_{\mathbf{m},\delta_0}$ denote the random graph which is constructed on the vertex set $\{1, \dots, r\}$ by creating edges between $1 \leq i < j \leq r$ independently with probability

$$1 - \mathrm{e}^{-m_i m_j / (\sum_{k=1}^{r} m_k)^{\delta_0}}.$$

It is instructive to interpret $\mathbf{m}$ as mass distribution and $|\mathbf{m}| := \sum_{k=1}^{r} m_k$ as *total mass* of $\mathcal{I}_{\mathbf{m},\delta_0}$.

Lemma 3.20 ([1, Lemma 2.5])**.** *Let $\delta_0 \in (1, 2)$ and $\varrho \in (0, 1)$ such that*

$$18\varrho > 16 + \delta_0.$$

There exist $\zeta = \zeta(\delta_0, \varrho) > 0$ and $M_0(\delta_0, \varrho) > 0$ such that for all $\mathbf{m}$ with $|\mathbf{m}| \geq M_0$

$$\mathbb{P}\left(N_{|\mathbf{m}|^\varrho}(\mathcal{I}_{\mathbf{m},\delta_0}) > 1 \right) < |\mathbf{m}|^{-\zeta},$$

where $N_x(\mathcal{I}_{\mathbf{m},\delta_0})$ denotes the number of clusters $C \subset \mathcal{I}_{\mathbf{m},\delta}$ satisfying $\sum_{j \in C} m_j \geq x$.

Proof of Lemma 3.19. Fix an l-level box B. As will become clear below, configurations of points which are too dense or too sparse are disadvantageous for our calculation, hence we assume that the event S occurs for η, with

$$S := \left\{ \left(d^{d/2} \prod_{i=0}^{l} \sigma_i^d \right)^{\delta/\delta'} \leq \sum_{\mathbf{x}=(x,s): x \in B} \mathbb{1}\{s \leq s_*\} \leq \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right)^{1/\varrho} \right\}, \tag{27}$$

and ϱ chosen according to (26). Let us take a closer look at the configuration of edges and vertices inside B^k. We work under the distribution $P_{\eta'} := \mathbb{P}(\,\cdot\,|\eta')$ and restrict ourselves to vertex configurations η' in S for the moment. Note that we may realise $\mathcal{G}_{|B^k}$ in a 2-stage sampling $((B^k), \emptyset) \to \mathcal{G}_1 \to \mathcal{G}_2 = \mathcal{G}_{|B^k}$, where first the edges in $\mathcal{G}_1$ are sampled independently with probability $p^1_{\mathbf{xy}}$, then edges are added independently at non-adjacent pairs in $\mathcal{G}_1$ with probability $p^2_{\mathbf{xy}}$ to obtain $\mathcal{G}_2$, precisely if $(p^i_{\mathbf{xy}}), i = 1, 2$, are chosen such that

$$\varphi(\mathbf{x}, \mathbf{y}) = p^1_{\mathbf{xy}} + (1 - p^1_{\mathbf{xy}}) p^2_{\mathbf{xy}}, \quad \mathbf{x}, \mathbf{y} \in B^k. \tag{28}$$

Now set

$$p^2_{\mathbf{x},\mathbf{y}} = 1 - \mathrm{e}^{\mathbb{1}\{\mathbf{x},\mathbf{y} \in B,\, s,t \leq s_*\} |x-y|^{-\delta d}}, \ \mathbf{x} = (x, s), \mathbf{y} = (y, t) \in B^k,$$

and define $(p^1_{\mathbf{xy}}, \mathbf{x}, \mathbf{y} \in B^k)$ via (28). Now sample $\mathcal{G}_1$ and recall that we would like to investigate the event $E = \{B \text{ is healthy}\}$. Note that E^c is the event that there are at least two $(l-1)$-level $(k, \prod_{j=0}^{l-1} \theta_j \sigma_j^d)$-preclusters inside B which are not connected in B^k. Consider all old vertices inside B, and denote them by $\mathbf{x}_1, \dots, \mathbf{x}_M$. Furthermore, denote by $B_1, \dots B_h$ the healthy sub-boxes of B and let $C_1^j, \dots C_{r(j)}^j$ denote the (maximal) semi-clusters of $\mathcal{G}_1$ inside sub-box B_j containing old vertices. Assume that $h \geq 1$ and $\sum_{j=1}^h r(j) \geq 2$, otherwise there is nothing to show. Let, furthermore, for any such cluster C, $m(C)$ denote the number of old vertices in it. We are now considering *only* connections between vertices which are sampled in obtaining $\mathcal{G}_2$ from $\mathcal{G}_1$. $\mathbf{x}, \mathbf{y} \in \{\mathbf{x}_1, \dots, \mathbf{x}_M\}$, which are not yet joined by an edge, become connected independently with probability

$$p^2_{\mathbf{x},\mathbf{y}} = 1 - \mathrm{e}^{|x-y|^{-\delta d}}, \ \mathbf{x}, \mathbf{y} \in \{\mathbf{x}_1, \dots, \mathbf{x}_M\}.$$

The independent edges may connect clusters $C, D \in \mathfrak{C} = \{C_i^j, 1 \leq i \leq r(j), 1 \leq j \leq h\}$. Since the clusters are disjoint this occurs independently with probability at least

$$1 - \mathrm{e}^{m(C)m(D)|\sqrt{d}m_l|^{-\delta d}} \geq 1 - \mathrm{e}^{m(C)m(D)M^{-\delta'}}, \quad C, D \in \mathfrak{C}$$

where $m_l = \prod_{i=0}^l \sigma_i$ and the inequality is due to (27) and our assumption that S has occurred. Using that $\sum_{C \in \mathfrak{C}} m(C) = M$, it is easily seen that the cluster-joining mechanism dominates an inhomogeneous random graph with parameter δ' and mass distribution $\mathbf{m} = (m(C), C \in \mathfrak{C})$. Thus, since $\varrho > (16 + \delta')/18$ we may apply Lemma 3.20 to deduce the existence of $\zeta = \zeta(\varrho, \delta')$ such that, for all $M \geq M_0(\varrho, \delta')$, the probability that $\mathcal{G}_2$ contains more than one cluster containing old vertices of size at least M^ϱ is at most $M^{-\zeta}$. Note that this estimate does not depend on the position of the vertices inside B^k. Now recall that on S,

$$M^\varrho < s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d$$

and hence there cannot be two distinct clusters of size $s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d$ in B^k up to an error of order

$$M^{-\zeta} \leq \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right)^{-\zeta} =: \varepsilon_l,$$

i.e.

$$\mathbb{1}_S P_{\eta'}(B \text{ is healthy}) \geq \mathbb{1}_S \left(1 - \varepsilon_l\right). \tag{29}$$

It remains to estimate $\mathbb{P}(S)$. Note that $Z := \sum_{(x,s) \in B} \mathbb{1}\{s \leq s_*\}$ is a Poisson random variable with parameter $\mu := s_* \prod_{i=0}^{l-1} \sigma_i^d$. A standard tail estimate for Poisson r.v. such as [21, Theorem 5.4] tells us that, for μ sufficiently large,

$$\mathbb{P}\left(Z < \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right)^{1/\varrho} \right) < \mathrm{e}^{-\mu(1-2/\mathrm{e})}$$

and

$$\mathbb{P}\left(Z > s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right) \leq \mathrm{e}^{-\mu}$$

and consequently

$$\mathbb{P}(S^c) \leq 2\mathrm{e}^{-(1-2/\mathrm{e}) s_* \prod_{i=0}^{l-1} \sigma_i^d}.$$

Combining this with (29), we obtain

$$\begin{aligned} h_l &\geq \mathbb{E}[\mathbb{1}_S P_{\eta'}(B \text{ is healthy})] \\ &\geq 1 - \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d \right)^{-\zeta} - 4\mathrm{e}^{-(1-2/\mathrm{e}) s_* \prod_{i=0}^{l-1} \sigma_i^d}. \end{aligned}$$

□

We now consider the boxes $B_{m_l}, l \geq 0$, which have 0 as their origin.

Lemma 3.21. *Fix $a, b > 0$ such that*

$$b < \left(1 - \frac{\delta}{\delta'}\right) da, \tag{30}$$

and set $\theta_l = (\bar{l} + l)^{-b}, l \geq 1$ and $\sigma_l = (\tilde{l} + l)^a, l \geq 0$, where $\tilde{l}, \bar{l} \in \mathbb{N}$ are fixed. It holds that

$$\mathbb{P}(\exists l \in \mathbb{N} : B_{m_l} \textit{ is not healthy or not alive}) \leq f(\tilde{l}, \bar{l})$$

where $f : \mathbb{N}^2 \to [0, 1]$ is such that for all sufficiently large $\bar{l}$, $\lim_{\tilde{l}\to\infty} f(\tilde{l}, \bar{l}) = 0$.

Proof. Note that by the choice of a, b the condition (25) can be satisfied for $\tilde{l}, \bar{l}$ sufficiently large. By Lemma 3.18 and the fact that $1/(1 - \theta_l) \leq 1/(1 - \theta_0)$ for our choice of θ, we obtain that the probability in question is bounded by

$$\begin{aligned} &\mathbb{P}(B_{m_0} \text{ is not healthy}) + \sum_{l=1}^{\infty} \Big(\frac{1 - h_{l-1}}{1 - \theta_l} + 1 - h_l\Big) \\ &\leq \frac{2}{1 - \theta_0} \left(\mathbb{P}(B_{m_0} \text{ is not healthy}) + \sum_{j=1}^{\infty} (1 - h_j)\right). \end{aligned} \tag{31}$$

Note, that $1 - h_0 = \mathbb{P}(B_{m_0}$ is not healthy) can be made arbitrarily small by choosing $m_0 = \sigma_0 = \tilde{l}$ large enough. To estimate the series on the right of (31), we use Lemma 3.19 and our choice for σ, θ:

$$\begin{aligned} \sum_{l=1}^{\infty} (1 - h_l) &\leq \sum_{l=1}^{\infty} \left(s_* \prod_{i=0}^{l-1} \theta_i \sigma_i^d\right)^{-\zeta} + 4\mathrm{e}^{-(1-2/\mathrm{e})s_* \prod_{i=0}^{l-1} \sigma_i^d} \\ &\leq \sum_{l=0}^{\infty} \left(s_* \prod_{i=0}^{l} \theta_i \sigma_i^d\right)^{-\zeta} + \sum_{l=0}^{\infty} 4\mathrm{e}^{-(1-2/\mathrm{e})s_* (\sigma_0^d)^{l+1}} \end{aligned}$$

The right hand series clearly vanishes as $\sigma_0 = \tilde{l} \to \infty$. The left hand series is a multiple of

$$\sum_{l=0}^{\infty} \prod_{i=0}^{\infty} \frac{(\bar{l} + i)^{\zeta b}}{(\tilde{l} + i)^{\zeta ad}} = \frac{\bar{l}^{\zeta b}}{\tilde{l}^{\zeta ad}} \left(1 + \sum_{l=1}^{\infty} \prod_{i=1}^{l} \frac{(\bar{l} + i)^{\zeta b}}{(\tilde{l} + i)^{\zeta ad}}\right).$$

For given $\bar{l}$, both factors on the right hands side are decreasing in $\tilde{l}$ and the series convergences for any choice of $\tilde{l}$, since

$$\frac{(\bar{l} + i)^{\zeta b}}{(\tilde{l} + i)^{\zeta ad}} \approx i^{\zeta(b - ad)},$$

as $i \to \infty$ and $b - ad < 0$. It follows that, for fixed $\bar{l}$, the right hands side of (31) vanishes, as $\tilde{l}$ gets large. □

It remains to prove Proposition 3.15.

Proof of Proposition 3.15. Fix $\kappa \in (0, 1)$ and $a > 0$ and $b > 0$ such that

$$b < \min\left\{(1 - \kappa), \left(1 - \frac{\delta}{\delta'}\right)\right\} da,$$

and let $r(n) = \prod_{j=0}^{n} j^a$. Combining Lemma 3.21 with Proposition 3.17, we obtain that with probability exceeding $1 - \varepsilon$ we find in the box B_{m_n} a cluster of size

$$\prod_{j=0}^{n} (\min\{\bar{l}, \tilde{l}\} + j)^{da-b} = \left(\prod_{j=0}^{n} (\min\{\bar{l}, \tilde{l}\} + j)^a\right)^{da-b/a} \geq r(n)^{da-b/a} \geq m_n^{\kappa d}.$$

□

4. Recurrence

In this section, we prove the recurrence results of Theorem 1.1, which are formulated in the following proposition.

Proposition 4.1. *Let $d \in \{1, 2\}$ and consider the weight-dependent random connection model.*
(a) If h is the preferential attachment kernel, min kernel, or sum kernel and $\delta > 2$ and $\gamma < \frac{\delta}{1+\delta}$,
(b) or if h is the product kernel and $\delta > 2$ and $\gamma < \frac{1}{2}$,
then the infinite percolation cluster, if it exists, is recurrent.

To this end, we adapt results from Berger [1] to our generalised continuum setup, and relegate the proofs of the claims to Appendix A.

Lemma 4.2. *Let $\mathbf{X}_\infty$ be a unit intensity Poisson process on $\mathbb{R}$. Consider a random graph on this point process, where points $x, y \in \mathbf{X}_\infty$ are connected with probability $P_{|x-y|}$, such that*

$$\limsup_{v\to\infty} v^2 P_v < \infty.$$

Then any infinite component of this graph is recurrent.

Lemma 4.3. *Let $\mathbf{X}_\infty$ be a unit intensity Poisson process on $\mathbb{R}^2$. Consider a random graph on this point process, where points $x, y \in \mathbf{X}_\infty$ are connected with probability $P_{|x_1-y_1|,|x_2-y_2|}$, such that*

$$\limsup_{u,v\to\infty} (u+v)^4 P_{u,v} < \infty.$$

Then any infinite component of this graph is recurrent.

Mind that neither of the two lemmas above requires independence of the edge occupancies. With these two lemmas at hand, we can prove the recurrence result.

Proof of Proposition 4.1. We start by considering the preferential attachment kernel h^{pa}. We only have to verify that the condition of Lemma 4.3 (resp. Lemma 4.2) holds. By translation invariance of the edge distribution, we can without loss of generality just consider the distance between two points to be equal to v and then send v to ∞. Furthermore, we have due to (8) that there exists a constant c_ρ depending only on the choice of the function ρ, such that the connection probability (for $t > s$) can be upper bound by

$$\rho(s^\gamma t^{1-\gamma} v^d) \le \begin{cases} c_\rho (s^\gamma t^{1-\gamma} v^d)^{-\delta} & \text{if } t > v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} \\ 1 & \text{otherwise.} \end{cases} \tag{32}$$

We start by considering the case $\gamma \le \frac{1}{2}$. In this case, since $\delta \ge 1$, we have $1 - \delta(1-\gamma) < 0$. Therefore, we can write $\limsup_{v\to\infty} v^{2d} \mathbb{P}(\text{two points at distance } v \text{ are connected})$ as

$$\begin{aligned}
&\limsup_{v\to\infty} v^{2d} \int_0^1 ds \int_s^1 \varphi\big((0,s),(v,t)\big) dt \\
&\le \limsup_{v\to\infty} v^{2d} c_\rho \int_0^1 ds \Big\{ v^{-\delta d} s^{-\gamma\delta} \int_{s\vee(v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)}\wedge 1)}^1 t^{-\delta(1-\gamma)} dt \\
&\qquad\qquad + \int_s^{s\vee(v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)}\wedge 1)} dt \Big\} \\
&= \limsup_{v\to\infty} v^{2d} c_\rho \int_0^1 ds \Big\{ v^{-\delta d} s^{-\gamma\delta} \Big(\frac{1 - (s \vee (v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} \wedge 1))^{1-\delta(1-\gamma)}}{1-\delta(1-\gamma)} \Big) \qquad (33) \\
&\qquad\qquad + s \vee (v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} \wedge 1) - s \Big\}, \qquad (34)
\end{aligned}$$

where we have in the second step broken the integral into two parts according to (32). We now consider three cases:

$$s \vee (v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} \wedge 1) = \begin{cases} 1 & \text{for } s \in (0, v^{-d/\gamma}), \\ v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} & \text{for } s \in (v^{-d/\gamma}, v^{-d}), \\ s & \text{for } s \in (v^{-d}, 1). \end{cases}$$

Therefore, (34) can be written as

$$\limsup_{v\to\infty} v^{2d} c_\rho \Big\{ \int_0^{v^{-d/\gamma}} (1-s)ds + \int_{v^{-d}}^1 \frac{v^{-\delta d} s^{-\delta\gamma+1-\delta(1-\gamma)}}{\delta(1-\gamma)-1} ds \tag{35}$$

$$+ \int_{v^{-d/\gamma}}^{v^d} \Big(\frac{v^{-d\gamma - \frac{d}{1-\gamma}(1-\delta(1-\gamma)} s^{-\gamma\delta - \frac{\gamma}{1-\gamma}(1-\delta(1-\gamma))}}{\delta(1-\gamma)-1} + v^{-d/(1-\gamma)} s^{-\gamma/(1-\gamma)} - s \Big) ds \tag{36}$$

$$+ \int_{v^{-d/\gamma}}^1 \frac{v^{-\delta d} s^{-\gamma\delta}}{1-\delta(1-\gamma)} ds \Big\}. \tag{37}$$

The integrals in (35) and (36) are all by a straightforward integration of order at most v^{-2d}. For the integral in (37) we must treat separately the cases $\gamma\delta < 1$, $\gamma\delta = 1$ and $\gamma\delta > 1$. Since $1 - \delta(1-\gamma) < 0$, we get that either the expression is positive and tends to 0 or is strictly negative.

The case $\gamma \in (\frac{1}{2}, \frac{\delta}{\delta+1})$ follows the same steps and after integrating and using that $\gamma \in (\frac{1}{2}, \frac{\delta}{\delta+1})$, one similarly obtains that all terms vanish as $v \to \infty$. The proof for the min (and by extension the sum) kernels is likewise similar, with calculations that are noticeably easier due to the simplified form of the kernel in question. Finally, the result for the product kernel is proven following the same steps, as shown for the lattice model in [11] $\square$

Appendix A. Auxiliary results

In this section, we demonstrate the proof of Lemmas 4.2 and 4.3. For this purpose we adapt the arguments of Berger [1] from the lattice to the continuum case. As in [1], the proof of Lemma 4.2 relies on the result of Nash and Williams, which we state here for completeness.

Theorem A.1 (Nash-Williams, [25]). *Let G be a graph with conductance C_e on every edge e. Consider a random walk on the graph such that when the particle is at some vertex, it chooses its way with probabilities proportional to the conductances on the edges that it sees. Let $\{\Pi_n\}_{n=1}^\infty$ be disjoint cut-sets, and denote by C_{Π_n} the sum of the conductances of Π_n. If*

$$\sum_n C_{\Pi_n}^{-1} = \infty,$$

then the random walk is recurrent.

In order to apply the theorem, we will adapt the definitions and lemmas from [1] to our setting where needed, starting with the following.

Definition A.2 (Continuum Bond Model). Let β be such that

$$\int_0^1 \int_k^{k+1} \beta(x-y)^{-2} dy dx > P_k$$

for every k. The *continuum bond model* is the two dimensional inhomogeneous Poisson process ξ with density $\beta(x-y)^{-2}$. We say that two sets A and B are connected if $\xi(A \times B) > 0$.

As observed in [1], if I is an interval and M is the length of the shortest interval that contains all of the vertices that are directly connected to I in the original model and if M' is the length of the smallest interval J such that $\xi(I \times (\mathbb{R}\backslash J)) = 0$, then M' stochastically dominates M. Since in our case both models are defined on a Poisson point process, this claim remains valid.

Lemma A.3. *(a) Under the conditions of Lemma 4.2, let I be an interval of length N. Then, the probability that there exists a vertex of distance bigger than d from the interval, that is directly connected to the interval, is $O(\frac{N}{d})$.*
(b) Consider the continuum bond model I be an interval of length N, and let J be the smallest interval such that $\xi(I \times (\mathbb{R}\backslash J)) = 0$. Then $\mathbb{P}(|J| > d) = O(\frac{N}{d})$.

Proof. (a) Let $\beta' = \sup_x \frac{P_x}{x^2} < \infty$. For vertex v at distance d from I, the probability that it is directly connected to I is bounded by

$$\mathbb{E}\Big[\sum_{\mathbf{X}_\infty \cap [0,N]} P_{d+|x|}\Big] \leq \mathbb{E}\Big[\beta' \sum_{\mathbf{X}_\infty \cap [0,N]} (d+|x|)^{-2}\Big] = \beta' \int_d^{d+N} y^{-2} dy < \frac{\beta' N}{d^2},$$

where we used Campbell's formula in the second step of the calculation. A second application yields that the probability of a vertex at distance bigger than d that is directly connected to I is bounded by

$$\mathbb{E}\Big[\sum_{\mathbf{X}_\infty \cap [N+d,\infty]} \frac{\beta' N}{|x|^2}\Big] \leq \int_d^\infty \frac{\beta' N}{|x|^2} dx = O\Big(\frac{N}{d}\Big).$$

The proof of (b) follows the same lines. $\square$

Lemma A.4. *Under the conditions of Lemma 4.2, for an interval I of length N, the expected number of open edges exiting I is $O(\log N)$. Furthermore, there exists a constant C_1, such that the probability of having more than $C_1 \log N$ open edges exiting I is smaller than $1/2$.*

Proof. As before, let $\beta' = \sup_x \frac{P_x}{x^2} < \infty$. For an interval $I = [a,b]$ with $|b-a| > 2$, we define the "1-interior" $\mathring{I}$ of I via $\mathring{I} = [a+1, b-1]$. Then, the expected number of open edges exiting I can be bound by

$$\begin{aligned}\mathbb{E}\Big[\sum_{\substack{u \in \mathbf{X}_\infty \cap \mathring{I} \\ v \in \mathbf{X}_\infty \cap I^c}} \mathbb{P}(u \leftrightarrow v)\Big] &\leq \beta' \mathbb{E}\Big[\sum_{\substack{u \in \mathbf{X}_\infty \cap \mathring{I} \\ v \in \mathbf{X}_\infty \cap I^c}} (u-v)^{-2}\Big] = \beta' \int_{\mathring{I}} \int_{I^c} (u-v)^{-2} dv\, du \\ &= 2\beta' \int_1^{N-1} \int_x^\infty y^{-2} dy\, dx \leq 4\beta' \int_1^{N-1} \frac{1}{x} dx = O(\log N), \end{aligned} \tag{38}$$

where in the second step we used (analogously to Campbell's formula) that the second moment measure of a Poisson point process on disjoint intervals corresponds to the Borel measure $dvdu$. To bound the expected number of edges exiting I that begin in $I\backslash\mathring{I}$, note that we can assume that all edges of length 1 or less are deterministically open. Therefore, we obtain the bound

$$\mathbb{E}\Big[\sum_{\substack{u \in \mathbf{X}_\infty \cap (I\backslash\mathring{I}) \\ v \in \mathbf{X}_\infty \cap I^c}} \mathbb{P}(u \leftrightarrow v)\Big] \leq 2\int_0^1 \int_x^{x+1} dy dx + 2\beta' \int_0^1 \int_{x+1}^\infty y^{-2} dy dx = O(1),$$

which combined with (38) proves the first claim.

To prove the second claim, let C be a constant large enough that the expected number of open edges exiting I is smaller than $C \log N$ for all N. Then, if $C_1 > 2C$, using Markov's inequality gives that the probability of more than $C_1 \log N$ open edges exiting I is smaller than $1/2$. $\square$

Lemma A.5 ([1, Lemma 3.7])**.** *Let A_i be independent events such that $\mathbb{P}(A_i) \geq 1/2$ for every i. Then*

$$\sum_{i=1}^\infty \frac{1_{A_i}}{i} = \infty \quad \textit{almost surely.}$$

Lemma 4.2 now follows by the same argument as in [1], which we restate here for completeness.

Proof of Lemma 4.2. Our goal is to show that almost surely, the infinite cluster satisfies the Nash-Williams condition from Theorem A.1 . We begin with some fixed interval I_0 and define I_n

inductively to be the shortest interval containing I_{n-1} and all of the vertices that are connected directly to I_{n-1}. We also define $D_n = \frac{|I_{n+1}|}{|I_n|}$.

Due to the translation invariance of the probability space, the edges exiting I_{n+1} are stochastically dominated by the edges exiting an interval of length $|I_{n+1}|$. In addition, by construction and given I_n, the edges exiting I_{n+1} are independent of the edges exiting I_n. Let now $\{U_n\}_{n=1}^{\infty}$ be independent copies of the continuum model as defined in Definition A.2. Then, the sequence D_n is stochastically dominated by the sequence $D'_n = \frac{|I'_{n+1}|}{|I_n|}$, where I'_{n+1} is the smallest interval such that $\mathbb{R}\backslash I'_{n+1}$ is not connected to the copy of I_n in U_n.

By definition, the variables D'_n are i.i.d. Therefore, by Lemma A.3 the sequence $\{\log(D_n)\}$ is dominated by the sequence of i.i.d. variables $d_n := \log(D'_n)$, for which it holds that $\mathbb{E}(d_n) < M$ for some finite constant M. Let Π_n be the collection of edges exiting I_n. Then, by construction $\{\Pi_n\}_{n=1}^{\infty}$ are disjoint cut-sets. Furthermore, given the intervals $\{I_n\}_{n=1}^{\infty}$, the cut-set Π_N is independent of $\{\Pi_n\}_{n=1}^{N-1}$. By Lemma A.4, it holds for each N independently and with probability greater than 1/2 that

$$|\Pi_N| \leq C_1 \sum_{n=1}^{N} d_n. \tag{39}$$

By the strong law of large numbers, with probability 1, for all large N, it holds that

$$\sum_{n=1}^{N} d_n < 2MN. \tag{40}$$

Using (39) and (40) gives that

$$C_{\Pi_N}^{-1} > \frac{1}{2MN}$$

for each N with probability greater than 1/2 and therefore by Lemma A.5 the Nash-Williams condition is almost surely satisfied, which proves the claim. □

We now proceed to the proof of Lemma 4.3. We begin by considering the infinite component of our graph as an electrical network where all open edges have conductance 1. Next, we project the vertices of the graph onto $\mathbb{Z}^2$ and assign conductances to the nearest neighbour edges of the lattice that form a path between the given pair of vertices. More precisely, we construct the electrical network as follows:

(A) Add an edge between every two vertices (x_1, y_1) and (x_2, y_2) for which $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$ and $\lfloor y_1 \rfloor = \lfloor y_2 \rfloor$.
(B) Assign conductance 1 to every edge.

Note that by construction, this electrical network has higher effective conductance than the electrical network without the added edges from (A). We now map this electrical network onto the lattice $\mathbb{Z}^2$ using the following rule.

(C) Map every edge between (x_1, y_1) and (x_2, y_2) to the edge connecting $(\lfloor x_1 \rfloor, \lfloor y_1 \rfloor)$ and $(\lfloor x_2 \rfloor, \lfloor y_2 \rfloor)$. For each original edge, increase the conductance of the new edge by 1.
(D) *Glue* vertices (x_1, y_1) and (x_2, y_2) for which $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$ and $\lfloor y_1 \rfloor = \lfloor y_2 \rfloor$ together and map them to the vertex $(\lfloor x_1 \rfloor, \lfloor y_1 \rfloor)$. Furthermore remove the corresponding edge between the two vertices, i.e. the edge between (x_1, y_1) and (x_2, y_2) is not mapped to any edge.

Due to the *parallel law* for electrical networks, step (C) does not decrease the effective conductance of the network. Similarly for step (D), the act of glueing neighbouring vertices with a single edge of conductance 1 between them does not decrease the effective conductance. We now proceed to the final stage of the procedure, by projecting long edges onto nearest neighbour edges:

(E) For every long (i.e. not nearest neighbour) edge between (x_1, y_1) and (x_2, y_2) with conductance c_e we remove the edge and increase the conductance to each nearest neighbour bond in $[(x_1, y_1), (x_1, y_2)] \cup [(x_1, y_2), (x_2, y_2)]$ by $c_e(|x_1 - x_2| + |y_1 - y_2|)$.

Observe that due to the *serial law* for electrical networks, step (E) does not decrease the effective conductance of the network.

Lemma A.6. *Let $P_{x,y}$ satisfy the conditions of Lemma 4.3. For the electrical network G constructed in steps (A)-(E) it holds that*

(1) A.s. all conductances are finite.

(2) The effective conductance of the network is bigger or equal to the effective conductance of the original network.

(3) The distribution of the conductance of an edge in G is shift invariant.

(4) The conductance C_e of an edge has a Cauchy tail, i.e. there exists a constant c_1 such that $\mathbb{P}(C_e > c_1 n) \leq n^{-1}$ for every $n \geq 1$.

Proof. Claims 2 and 3 clearly hold by construction. To prove Claim 1, we calculate the expected number of bonds that are projected on the edge $(x, y), (x, y+1)$. We can w.l.o.g. assume that the projected edge starts at some $(x, y_1 \leq y)$, continues through $(x, y_2 \geq y+1)$, and ends at some (x_1, y_2). Before we proceed with the calculation however, note that we have for $P_{a,b} = 1 \wedge (a+b)^{-4}$ that

$$\int_0^\infty dx \int_0^\infty dy \int_{-\infty}^\infty P_{x+y,z} dz = 2 \int_{c_b}^\infty dx \int_{c_b}^\infty dy \int_{c_b}^\infty \frac{1}{(x+y+z)^4} dz, \tag{41}$$

where $c_b \in (0,1)$ is a constant satisfying

$$\int_{c_b}^1 dx \int_{c_b}^1 dy \int_{c_b}^1 \Big(\frac{1}{(x+y+z)^4} - 1\Big) dz = \int_{\mathbb{R}_+^3 \setminus [c_b, \infty)^3} P_{x+y,z} dx dy dz.$$

Note that the representation in (41) actually holds for any $P_{a,b}$ satisfying the conditions of the lemma and not just the specific function we have used, as it holds for every pair a, b that $P_{a,b} \leq 1$ and therefore one can always find an appropriate constant $c_b > 0$. Write now $\sum_A$ for the (random) sum across all vertices of the original Poisson point process that are mapped to A. The expected number of projected edges is therefore

$$\begin{aligned} 2\mathbb{E}\Big[\sum_{y_1 \leq y, y_2 \geq y+1, x_1} P_{|y_2 - y_1|, |x_1 - x|}\Big] &= 2 \int_{-\infty}^{y+1} dy_1 \int_{y+1}^\infty dy_2 \int_{-\infty}^\infty P_{|y_2-y_1|,|x_1-x|} dx_1 \\ &\leq 4M \int_{c_b}^\infty dj \int_{c_b}^\infty dk \int_{c_b}^\infty \frac{1}{(k+j+h)^4} dh \leq 4M \int_{c_b}^\infty dl \int_{c_b}^\infty \frac{1}{(l+h)^3} dh \leq 4M \int_{c_b}^\infty \frac{1}{s^2} ds, \end{aligned}$$

where $M = \sup_{x,y} (x+y)^4 P_{x,y}$ and where we have in the first step used that the second moment measure of a Poisson point process on disjoint areas corresponds to the Borel measure. It follows that the number of projected edges is almost surely finite and since all such edges are almost surely finite, so is the total projected conductance for each edge.

Claim 4 follows from the same calculation. $\square$

In order to prove Lemma 4.3 we need a final result from [1], which we provide here without proof.

Theorem A.7 ([1, Theorem 3.9])**.** *Let G be a random electrical network on the lattice $\mathbb{Z}^2$, such that all of the edges have the same conductance distribution, and this distribution has a Cauchy tail. Then, almost surely, G is recurrent.*

Notice that we do not require any independence in the theorem.

Proof of Lemma 4.3. By the steps (A)-(E) and the remarks following each step, the conductance between two lattice points of the projected electrical network is bigger than the effective conductance between their preimages in the original graph in the continuum. Therefore, if the projected electrical network is recurrent, so is the original graph. By Lemma A.6, the conductances of the projected network have Cauchy tails and by Theorem A.7, this implies that the network is recurrent. $\square$

It remains to verify the domination claim in the proof of Proposition 3.14.

Lemma A.8. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be vertices with given positions $x_i, 1 \leq i \leq N$, and random weights $S_1, \ldots, S_N$. Let A denote the event that $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are connected in $\mathcal{G}^\beta$ and let $M = \sum_{i=1}^N \mathbb{1}\{S_i \leq q\}$. Then*

$$\mathbb{P}(M > k|A) \geq \mathbb{P}(M > k),$$

i.e. conditionally on A, M dominates an independent Bin(N, q) random variable.

Proof. Since the vertex positions are given, $\mathbb{1}_A$ is a decreasing function of the vertex weights $S_1, \ldots, S_N$ and of the edge weights $U_{ij}, \{i,j\} \in \{1, \ldots, N\}^{[2]}$. Let

$$F_0 = \sigma(U_{ij}, \{i,j\} \in \{1, \ldots, N\}^{[2]}),$$

and define inductively $F_j = \sigma(F_{j-1}, S_j), j = 1, \ldots, N$. Conditionally on A and the edge weights, we reveal the weights $S_1, S_2, \ldots, S_N$ one by one. In the j-th step, the conditional probability to encounter a small weight is

$$\frac{\mathbb{E}(\mathbb{1}\{S_j \leq q\}\mathbb{1}_A|F_{j-1})}{\mathbb{E}(\mathbb{1}_A|F_{j-1})} \geq \mathbb{P}(S_j \leq q)$$

by Lemma A.9. This implies that we can couple the $j-th$ reveal with an independent Bernoulli(q) r.v. B_j such that $B_j = 1$ whenever $S_j \leq q$ conditionally on F_{j-1}. It follows that M dominates $\bar{M} = \sum_{i=1}^N B_j$, which proves the claim. □

Lemma A.9. *Let $V_i, i \in I$ be a finite collection of i.i.d. Uniform$[0,1]$ r.v., and let A be a decreasing event, i.e. $\mathbb{1}_A$ is component-wise decreasing in $V_i, i \in I$. For disjoint subsets $I_1, I_2 \subset I$ and any set of numbers $x_j \in [0,1], j \in I_2$ it holds that*

$$\mathbb{E}\left(\mathbb{1}\{V_j \leq x_j, j \in I_2\}\mathbb{1}_A|V_i, i \in I_1\right) \geq \prod_{j\in I_2} \mathbb{P}(V_j \leq x_j)\, \mathbb{E}(\mathbb{1}_A|V_i, i \in I_1).$$

Proof. Given $V_i, i \in I_1$, the function

$$\mathbb{1}_A = \mathbb{1}_A(V_i, i \in I_1; V_j, j \in I_2; V_k, k \in I \setminus (I_1 \cup I_2))$$

is still a decreasing function of $V_j, j \in I_2$. Thus the FKG-inequality for the product distribution Uniform$[0,1]^{\otimes I_2}$ yields

$$\mathbb{E}\left(\mathbb{1}\{V_j \leq x_j, j \in I_2\}\mathbb{1}_A|V_i, i \in I_1\right) \geq \prod_{j\in I_2} \mathbb{E}(\mathbb{1}\{V_j \leq x_j\}|V_i, i \in I_1)\, \mathbb{E}(\mathbb{1}_A|V_i, i \in I_1),$$

and the claim follows by independence. □

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