28: Structure and Dynamics of Directed Scale-free Networks

Christian Mönch
Johannes Gutenberg-Universität Mainz

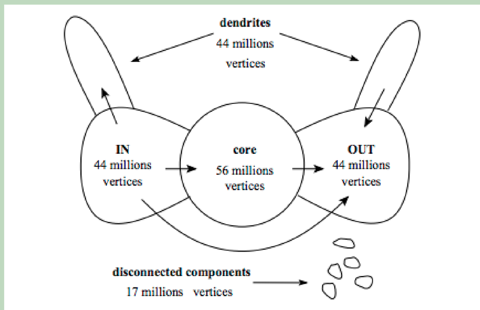
Applicant identifiers: MO 3831/2-1
Requested positions: Own positions: 1

Possible connections to projects: 14, 16, 17, 29

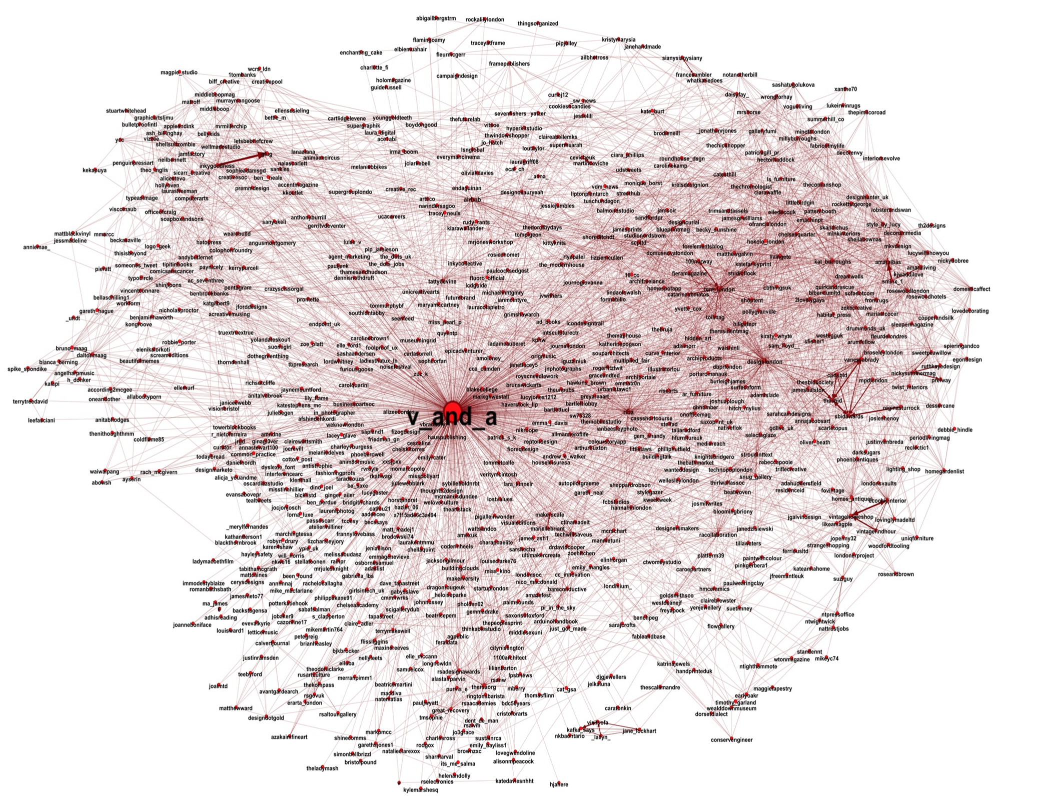
Sparse Scale-free Digraphs

- **Background**
  - ▹ Many real world networks are **directed**
  - ▹ Key examples: citation networks, follower networks in social media, financial networks, link-structure of WWW
  - ▹ Directed models mainly in **computer science** and **physics** literature
    - ∗ Numerical/data driven
    - ∗ Analytical heuristics
- **Modelling framework**
  - ▹ Random **finite digraphs** $G_N = (V_N, E_N)$, vertices V_N, directed edges / arcs E_N
  - ▹ System size parameter N – study large network **asymptotics** as $N \to \infty$
  - ▹ Focus on **sparse** networks with $O(|E_N|) = O(|V_N|)$
  - ▹ Important special case: **scale-free** graphs $\frac{1}{V_N}\sum_{v\in V_N} \mathbb{1}\{v \text{ has degree } k \text{ in } G_N\} = k^{-\tau+o(1)}$
  - ▹ Networks often dispay **clustering** → consider digraphs embedded in **space**

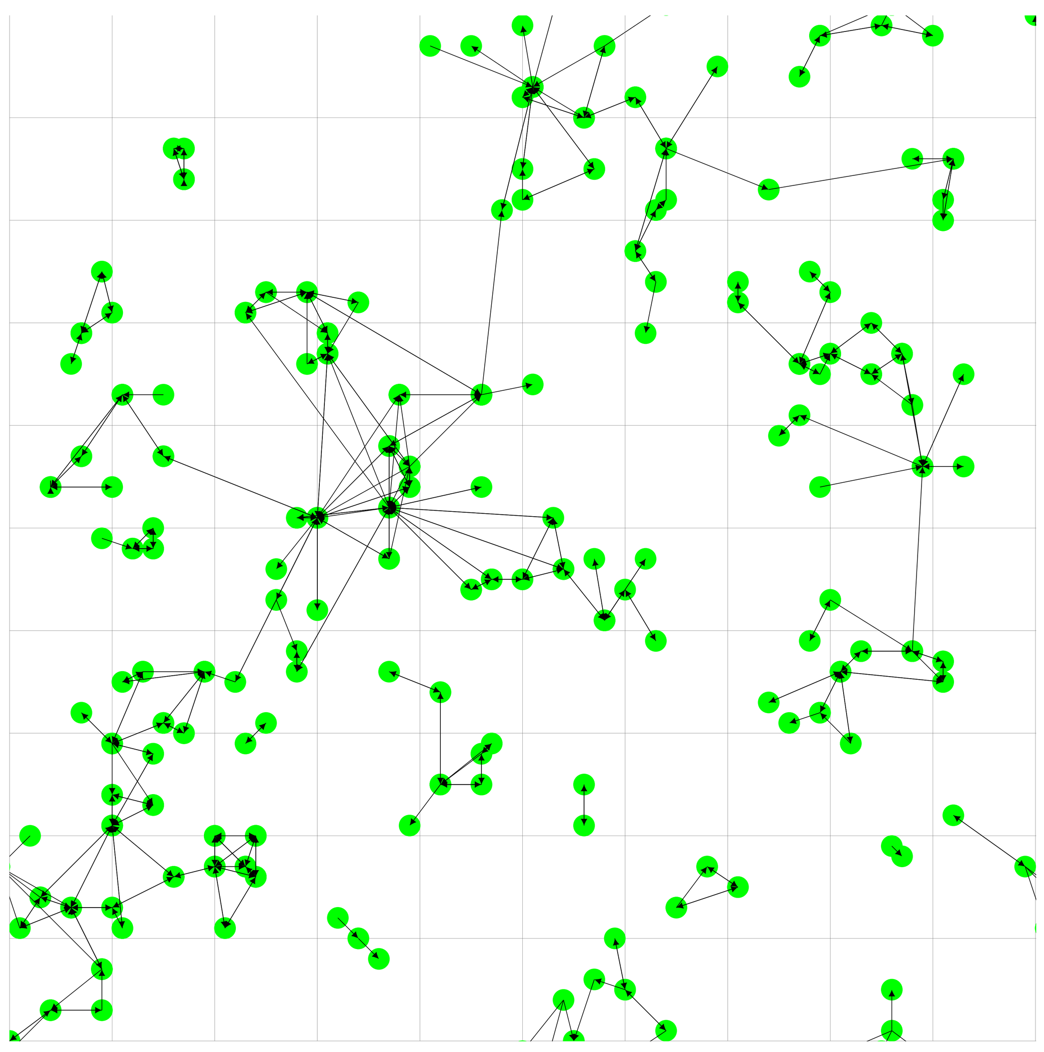
Rigorous mathematical results are scarce.



'Bowtie' structure of the WWW [1]



Twitter network of the Victoria and Albert Museum, London, in 2015 [2]


Directed random connection model with power law weights on the 2d-torus

Local vs. Global Structure / Power Laws vs. Geometry

- **Sketching possible models**
  - ▹ $V_N = \mathbb{Z}^d$ or **unit rate Poisson process** restricted to $[-N/2, N/2]^d$
  - ▹ Generate arcs **independently** with probability

$$\frac{\phi\,(U_v^{\rightarrow}, U_w^{\leftarrow})}{|v-w|^{\delta d}}, \quad v, w \in V_N$$

    - ∗ I.i.d. bivariate **weight sequence** $(U_v^{\leftarrow}, U_v^{\rightarrow})_{v\in V_N}$ generates local randomness and **in-/outdegree correlations**
    - ∗ **Profile** parameter $\delta > 1$ modulates effect of geometry
    - ∗ **Kernel** ϕ produces structural features – **power law degrees**, **preferential attachment**-like effects, etc.
  - ▹ Poisson model yields **Directed Random Connection Model with Weights** – directed version of [A]
  - ▹ Lattice model includes **Directed Scale-Free Percolation** – directed version of [B]
  - ▹ Combination of scale-free graphs with (long-range) percolation
- **Structural properties**
  - ▹ Local behaviour – **Large Deviation Theory**: strengthen the marked point process-LD approach of [C]
  - ▹ Global behaviour: percolation & robustness vs. targeted attacks, typical distances – adapt techniques from undirected setting e.g [A,B,D]

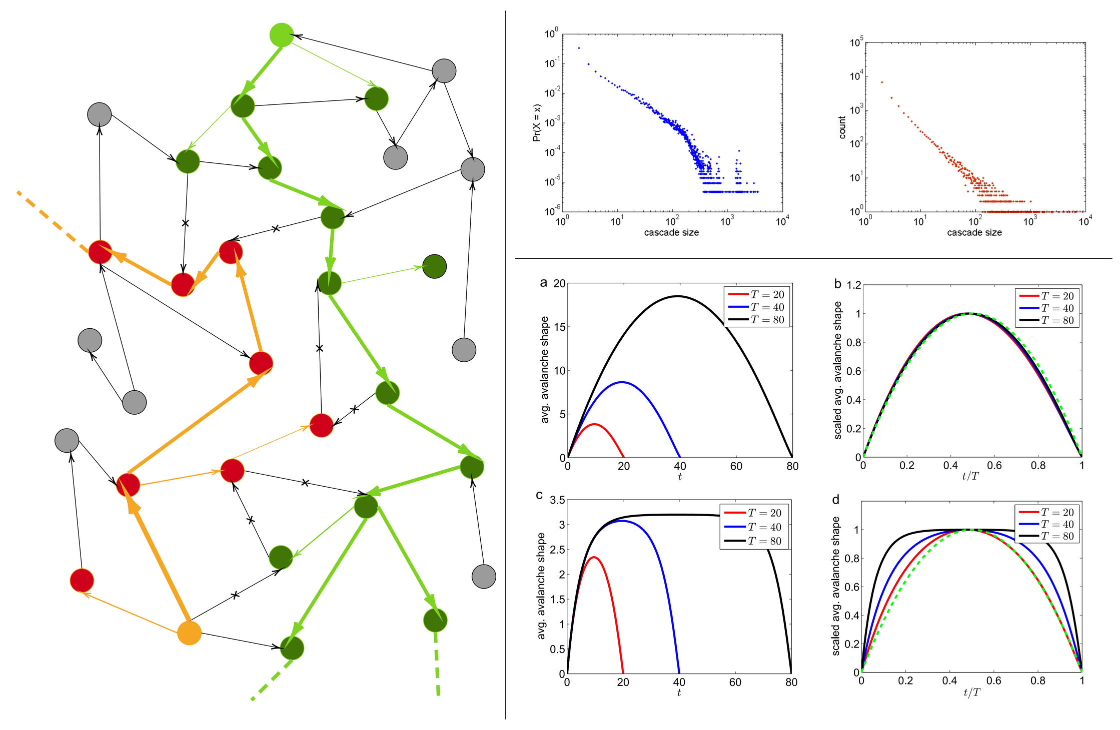
Spatial random digraphs are not locally treelike – they display realistic clustering.

Which properties are almost local?

Network Dynamics: Random Walk, Infection and Information

- **Random walk**
  - ▹ . . . on local limits – building on work in [A]
    - ∗ **Recurrence/transience** of supercritical strongly connected clusters in spatial models
    - ∗ **Invariance principles?** – also unknown for undirected scale-free setting
  - ▹ . . . on supercritical strongly connected clusters for large N: **mixing/cover times**
    - ∗ Directedness helps to gain independence, cf. [E]
    - ∗ "Spreading out" interpolates between spatial and non-spatial (di)graphs
- **SI-type dynamics/cascade models**
  - ▹ Multitude of models for spread of infection, rumours, accumulation of systemic risk
  - ▹ Example: **Threshold Contact Process** approximation to Boolean networks
    - ∗ Threshold dynamics are common in information diffusion and neural network models – transmission rates not proportional to number of infected neighbours
    - ∗ Particular interest on **critical** setting: rigorously verify exponents obtained in [F]

Focus on models requiring digraphs / display new features


Left: Stylised representation of competing SI-type dynamcis on a directed graph. *Right, top:* Empirical distribution of information cascade total sizes ('avalanches') in social networks Digg (blue) and Twitter (red) [3]. *Right, bottom:* Temporal avalanche shapes in idealised ODE model [4]. Top 2 images: critical dynamics – universality conjectured. Bottom 2 images: supercritical dynamics. RHS: profiles rescaled by total avalanche duration.

References

[A] P. Gracar, M. Heydenreich, C. Mönch and P. Mörters, *Recurrence versus Transience for Weight-Dependent Random Connection Models*, arXiv preprint arXiv:1911.04350 (2020+).

[B] M. Deijfen, R. van der Hofstad and G. Hooghiemstra, *Scale-free percolation*, Annales de l'IHP Probabilités et statistiques 49(3), pp. 817-838 (2013).

[C] C. Hirsch and C. Mönch, *Distances and large deviations in the spatial preferential attachment model*, Bernoulli, 26(2), pp. 927-947 (2020).

[D] M. Heydenreich, T. Hulshof and J. Jorritsma, *Structures in supercritical scale-free percolation*, The Annals of Applied Probability 27(4), pp. 2569-2604.

[E] C. Bordenave, P. Caputo and J. Salez, *Random walk on sparse random digraphs*, Probability Theory and Related Fields, 170(3-4), pp. 933-960 (2018).

[F] M. Möller and B. Drossel, *Scaling laws in critical random Boolean networks with general in-and out-degree distributions*, Physical Review E 87(5), 052106 (2013).

Image credits

[1] A. Broder et al., *Graph structure in the web*, Computer networks, 33(1-6), 309-320 (2000). The drawing is a reimaging of Fig. 9, p. 318.

[2] A. Espinos, *Museums on Twitter: Three case studies of the relationship between a museum and its environment*, MW2015: Museums and the Web 2015 (2015). Available via https://mw2015.museumsandtheweb.com/paper/museums-on-twitter-three-case-studies-of-the-relationship-between-a-museum-and-its-environment-museum-professionals-on-twitter/.

[3] K. Lerman and R. Ghosh, *Information contagion: An empirical study of the spread of news on Digg and Twitter social networks*, 4th International AAAI Conference on Weblogs and Social Media (2010). Image reproduced from preprint arXiv:1202.3162, Fig. 6.

[4] J. P. Gleeson and R. Durrett, *Temporal profiles of avalanches on networks*, Nature Communications 8:1227, pp.1-13 (2017). The image shown represents the Poisson setting, cf. Fig. 2, p.4.

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