A comment on the geometrical understanding of the Cauchy distribution

Jan Lukas Igelbrink & Anton Wakolbinger

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The standard Cauchy (or Cauchy(1))-distribution arises as the distribution of $\tan U$, where U is uniformly distributed on $(-\pi/2, \pi/2)$. It is well known that the sum of two independent standard Cauchy distributed random variables Z_1 and Z_2 has the same distribution as $2Z_1$. Cuadras [CU] remarks that *"... it is an open question to give a geometric but conscientious proof, elementary enough for teaching purposes, (i.e., without using double integrals or the characteristic function), that the mean of the tangents of uniform angles is the tangent of an angle also uniformly distributed in* $(-\pi/2, \pi/2)$*, i.e, following the Cauchy distribution."*

Stimulated by Cuadras' remark we will give a proof which we think has the attributes *geometric* and *conscientious*, and should also be adequate for (undergraduate) teaching purposes, at least for students having some knowledge of Brownian motion. We do not claim that our reasoning is new; it can certainly be extracted from the literature (see e.g. Sections 1.6 and 1.7 in [MC] and references therein). However, we think it might be good to have the proof written in a fairly self-contained manner. The argument uses Euler's formulae for the trigonometric functions and the fact that for a conformal mapping h and a planar Brownian motion B the process $(h(B_t))_{t\geq 0}$ is again a (time-changed) planar Brownian motion (see e.g. [L, Theorem 2.2]).

Proof. 1. We consider the Moebius transformation

$$h(z) := \frac{z-1}{z+1}, \quad z \in \mathbb{C}. \tag{1}$$

The map h transforms the unit circle S to the imaginary axis, with

$$h(e^{i\theta}) = \frac{e^{i\theta}-1}{e^{i\theta}+1} = i\frac{(e^{i\theta/2}-e^{-i\theta/2})/(2i)}{(e^{i\theta/2}+e^{-i\theta/2})/2} = i\tan\frac{\theta}{2},\ \theta \in (-\pi,\pi). \tag{2}$$

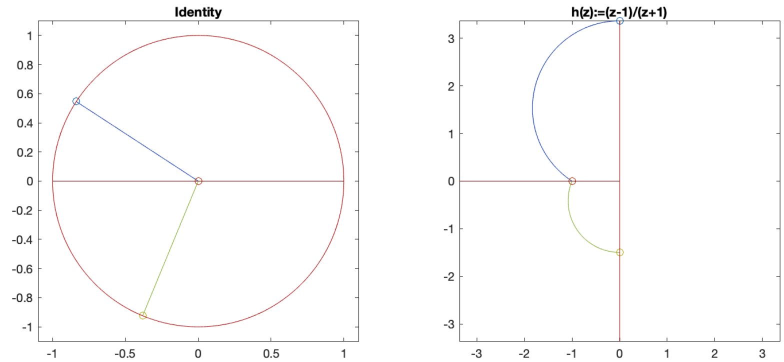

Figure 1: The conformal map h defined in (1) transforms straight line segments originating in $(0,0)$ and ending on S to circular arcs originating in $(-1,0)$ and ending on the y-axis.

We also note that h maps the origin to $(-1,0)$. (See Figure 1)

2. Let B be a standard planar Brownian motion (started in the origin) and h be as in 1. Write

$$R := \inf\left\{t \geq 0 : ||B_t||^2 = 1\right\} \tag{3}$$

for the random time at which B first hits S. By rotational invariance, B_R is uniformly distributed on S, hence by (2) the random variable $h(B_R)$ is Cauchy(1)-distributed on the y-axis. Because h is a conformal mapping, the process $(X,Y) := (h(B_t))_{t\geq 0}$ is, up to a time change depending only on the position of B, again a planar Brownian motion, now started in $(-1,0)$. Since h transforms the inner face of S to the half plane $\{z : \Re e(z) < 0\}$, the random time R equals the time T at which X first hits the point 0. This shows that the second component of a standard Brownian motion W started in $(0,0)$ is Cauchy distributed, when it is evaluated at the the time at which the first component of W first hits the point 1.

3. Let B' be an independent copy of B, let $(X',Y') := (h(B'_t))_{t\geq 0}$ and T' be the time at which X' first hits the point 0. Let

$$\widetilde{W}_t := \begin{cases} (1+X_t, Y_t) & \text{for } 0 \leq t \leq T \\ (2+X'_{t-T}, Y_T + Y'_{t-T}) & \text{for } T \leq t \leq T+T'. \end{cases}$$

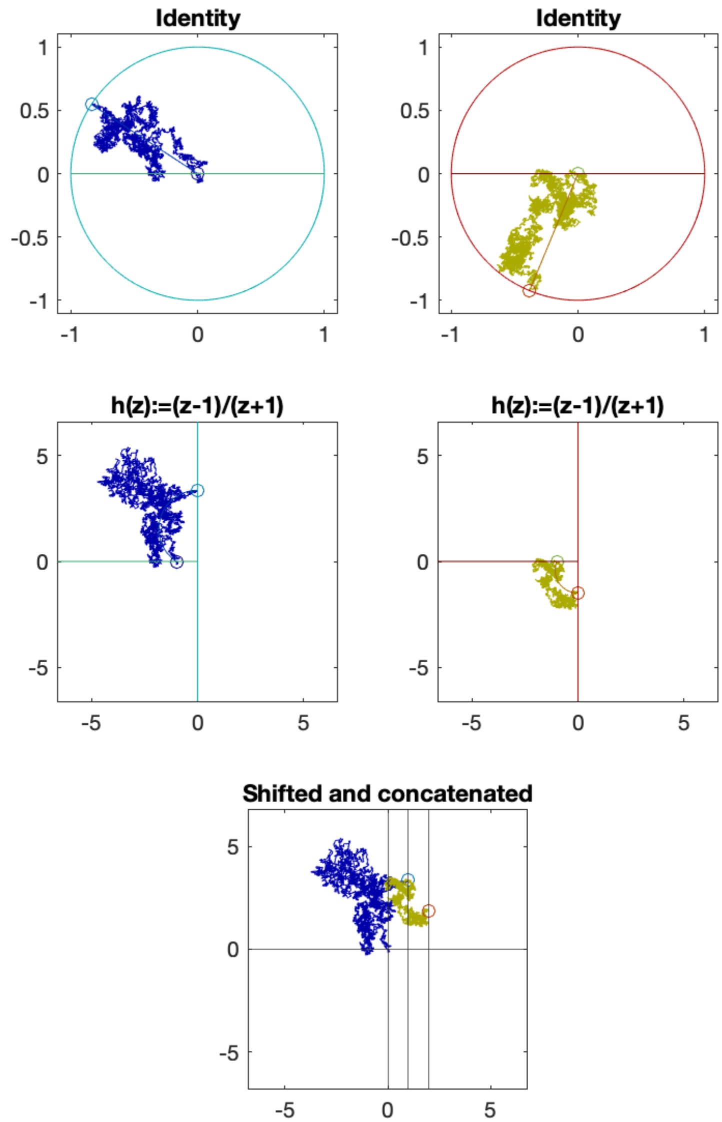

Figure 2: Two independent planar Brownian paths B, B'
a) stopped when they hit the unit circle (upper panel),
b) transformed with the mapping h defined in (1) (middle panel),
c) shifted and concatenated so that they result in a Brownian path W started in the origin and stopped when first hitting $\{x = 2\}$ (lower panel).

$\widetilde{W}$ is a time-changed planar Brownian motion starting in $(0,0)$, and $T+T'$ is the time at which its first component first hits 2. (See Figure 2 for an illustration how $\widetilde{W}$ arises from B and B'.) At the time $T+T'$, the second component of $\widetilde{W}$ equals $Y_T+Y'_{T'}$, and we know from step 2 that the independent random variables Y_T and $Y'_{T'}$ are Cauchy(1)-distributed.

4. Let $W=(W^{(1)},W^{(2)})$ be a standard Brownian motion started in the origin, and let T_i be the time at which $W^{(1)}$ first hits i, $i=1,2$. The process $\frac{1}{2}W$ is again a planar Brownian motion started in the origin, now with variance parameter $\frac{1}{4}$. Since T_2 is the time at which $\frac{1}{2}W^{(1)}$ first hits 1, we know from step 2 that $\frac{1}{2}W^{(2)}_{T_2}$ is Cauchy(1)-distributed. On the other hand we know from step 3 $W^{(2)}_{T_2}$ is distributed as $Y_T+Y'_{T'}$. $\square$

References

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